Important separators and spiders

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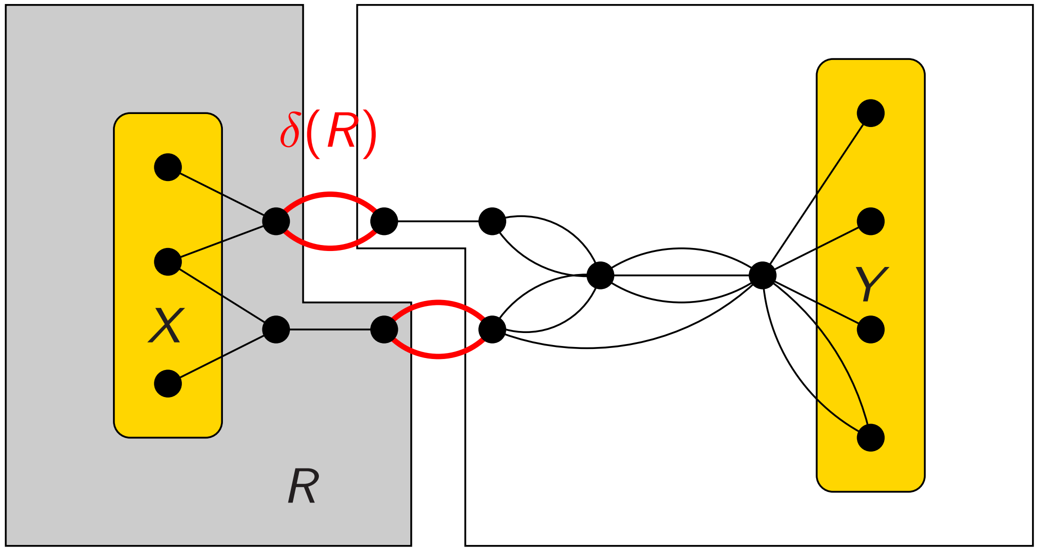
Overview

- Bounding the number of “important” separators.
- Two applications:
  - FPT algorithm for multiway cut.
  - Erdős-Pósa property for “spiders.”

Important separators

Definition: $\delta(R)$ is the set of edges with exactly one endpoint in R.

Definition: $\delta(R)$ is an (X, Y)-separator if $X \subseteq R$ and $R \cap Y = \emptyset$.



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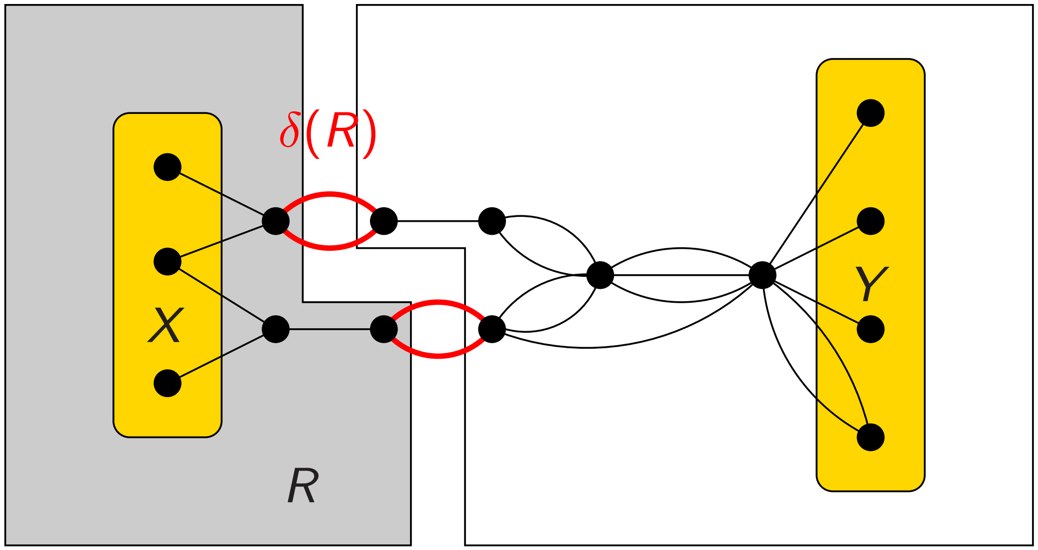
Note: Can be checked in polynomial time if a separator is important.



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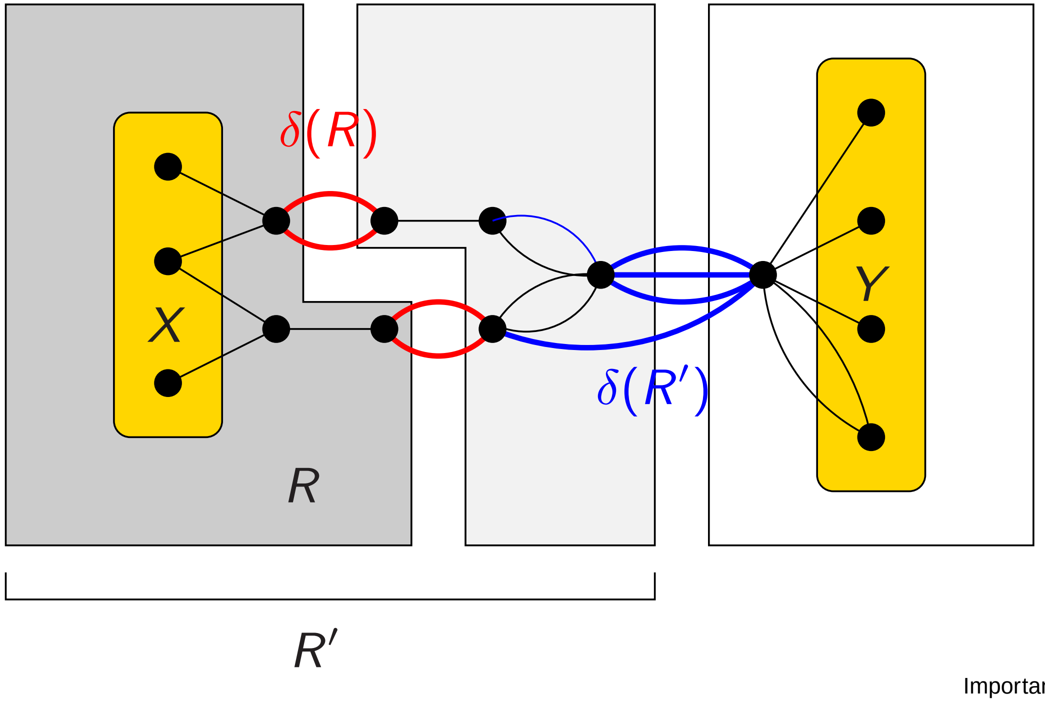

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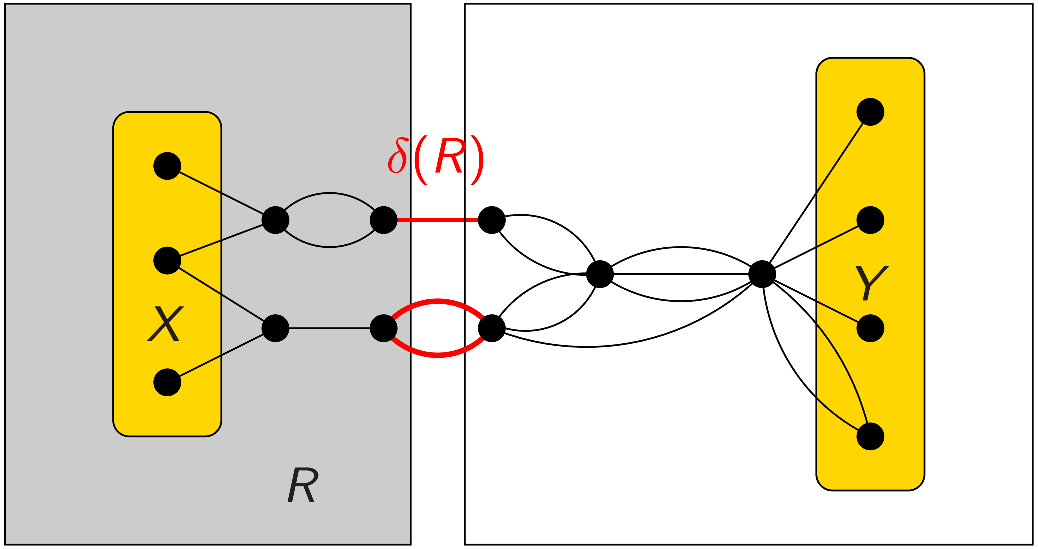
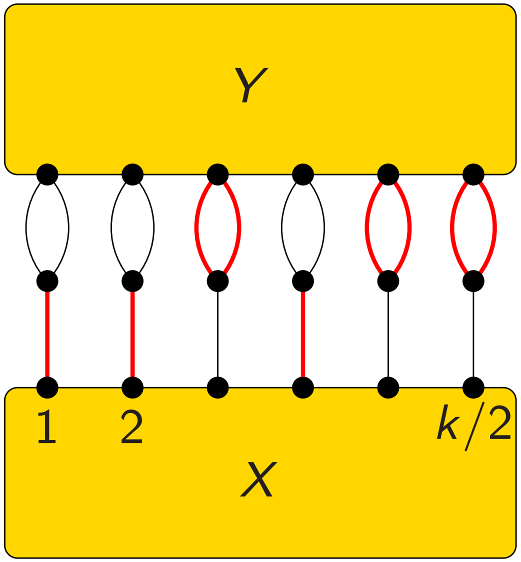


Important separators

The number of important separators can be exponentially large.

Example:



This graph has exactly $2^{k/2}$ important (X, Y)-separators of size at most k.

Theorem: There are at most 4^k important (X, Y)-separators of size at most k.
(Proof is implicit in [Chen, Liu, Lu 2007], worse bound in [M. 2004].)

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Proof: Let λ be the minimum (X, Y)-separator size and let $\delta(R_{\max})$ be the unique important separator of size λ and $R_{\max}$ is maximal.

First we show that $R_{\max} \subseteq R$ for every important separator $\delta(R)$.

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By the submodularity of δ:

$$\begin{array}{ccccccc} |\delta(R_{\max})| & + & |\delta(R)| & \geq & |\delta(R_{\max} \cap R)| & + & |\delta(R_{\max} \cup R)| \\ \lambda & & & & \geq \lambda & & \end{array}$$

$$\Downarrow$$

$$|\delta(R_{\max} \cup R)| \leq |\delta(R)|$$

$$\Downarrow$$

If $R \neq R_{\max} \cup R$, then $\delta(R)$ is not important.

Thus the important (X, Y)- and $(R_{\max}, Y)$-separators are the same.
$\Rightarrow$ We can assume $X = R_{\max}$.

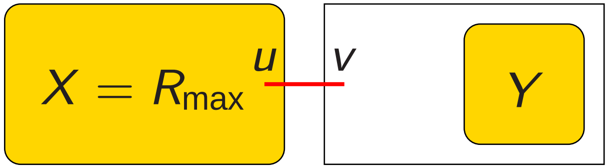
Important separators

Lemma: There are at most 4^k important (X, Y)-separators of size at most k.

Search tree algorithm for finding all these separators:

An (arbitrary) edge uv leaving $X = R_{\max}$ is either in the separator or not.

Branch 1: If $uv \in S$, then $S \setminus uv$ is an important (X, Y)-separator of size at most $k - 1$ in $G \setminus uv$.



Branch 2: If $uv \notin S$, then S is an important $(X \cup v, Y)$-separator of size at most k in G.

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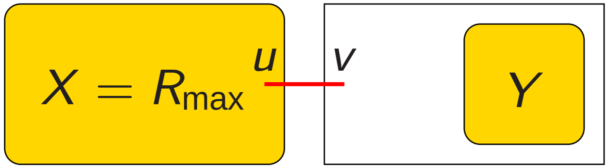
$\Rightarrow$ k decreases by one, λ decreases by at most 1.



Branch 2: If $uv \notin S$, then S is an important $(X \cup v, Y)$-separator of size at most k in G.

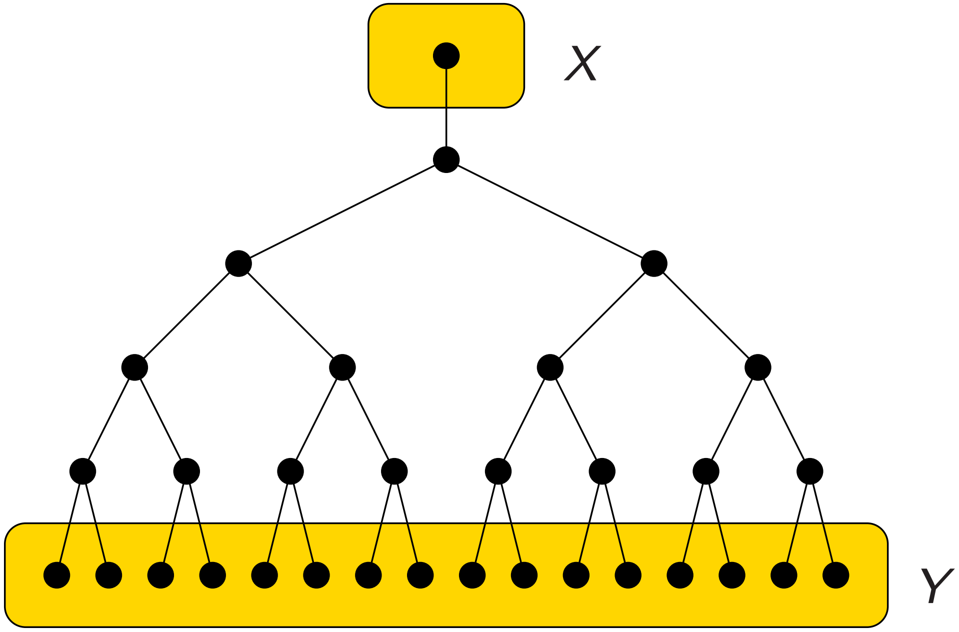
$\Rightarrow$ k remains the same, λ increases by 1.

The measure $2k - \lambda$ decreases in each step.

$\Rightarrow$ Height of the search tree $\leq 2k \Rightarrow \leq 2^{2k}$ important separators.

Important separators

Example: The bound 4^k is essentially tight.



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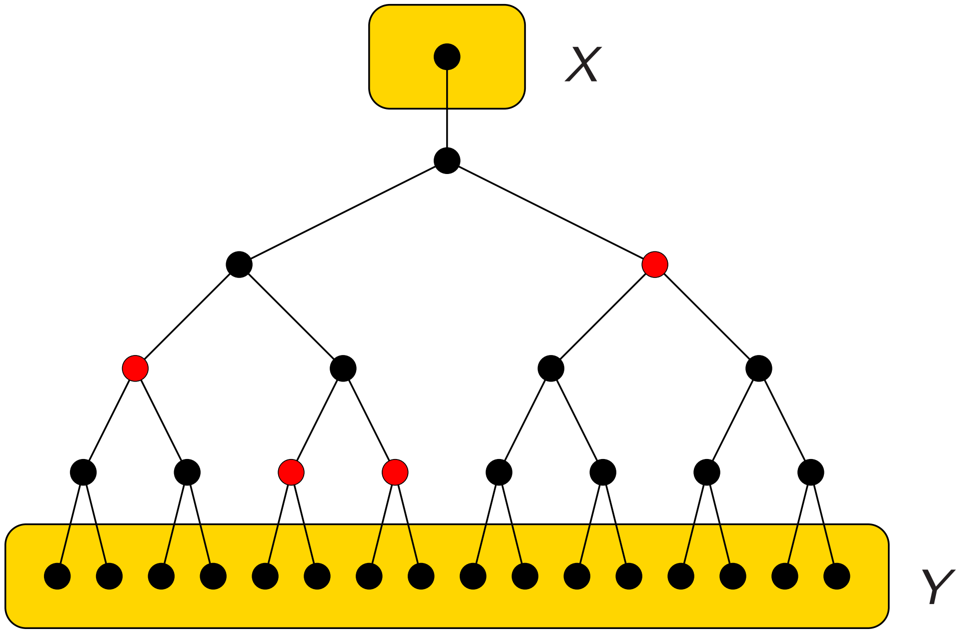

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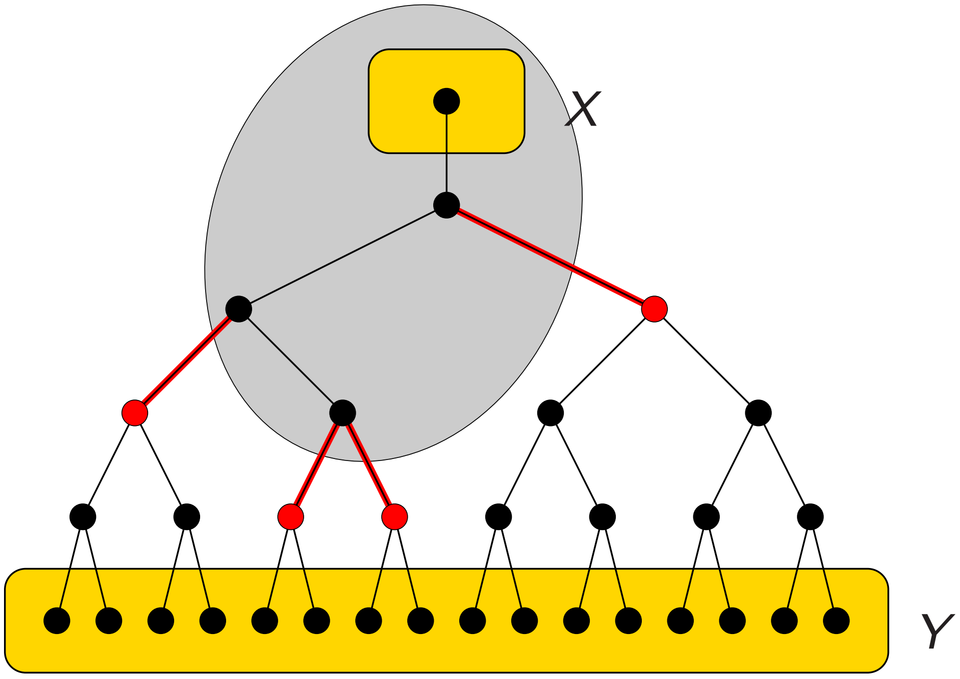

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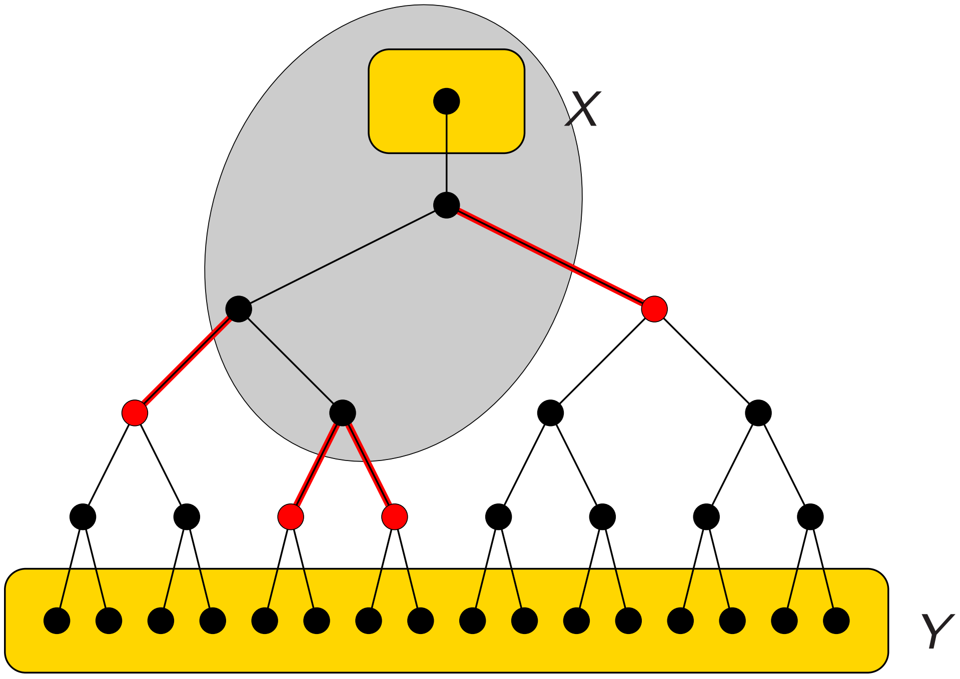

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The number of subtrees with k leaves is the Catalan number

$$C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1} \geq 4^k/\text{poly}(k).$$

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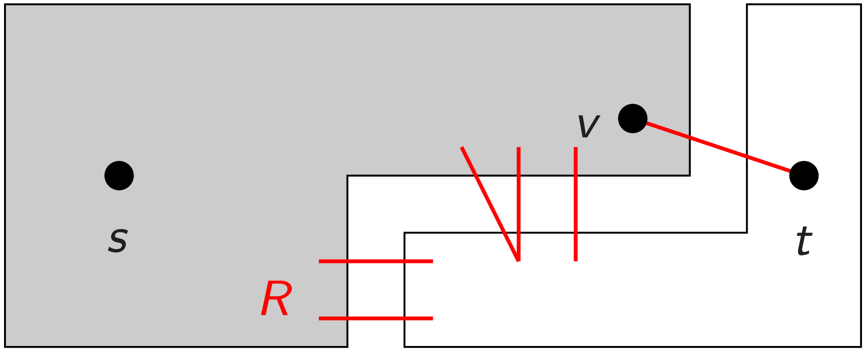

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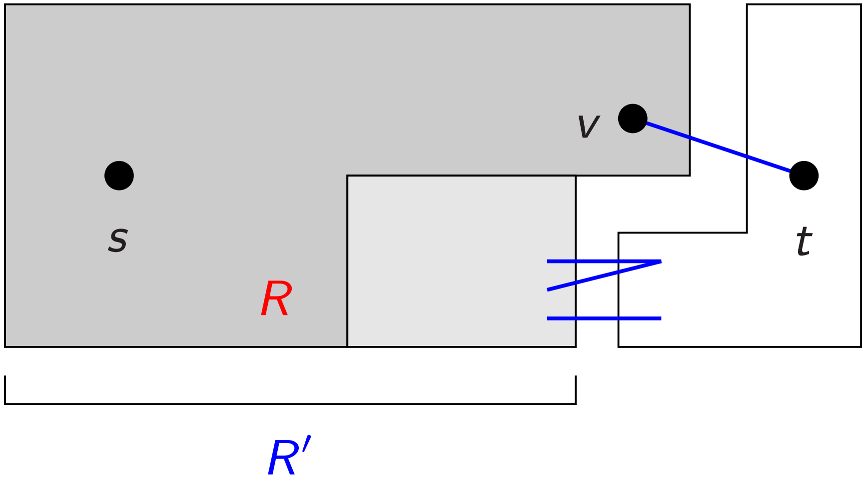


Suppose that $vt \in \delta(R)$ and $|\delta(R)| = k$.
There is an important (s,t)-separator $\delta(R')$ with $R \subseteq R'$ and $|\delta(R')| \leq k$.
Clearly, $vt \in \delta(R')$: $v \in R$, hence $v \in R'$.

MULTIWAY CUT

Task: Given a graph G, a set T of vertices, and an integer k, find a **multiway cut** S of at most k edges: each component of $G \setminus S$ contains at most one vertex of T.

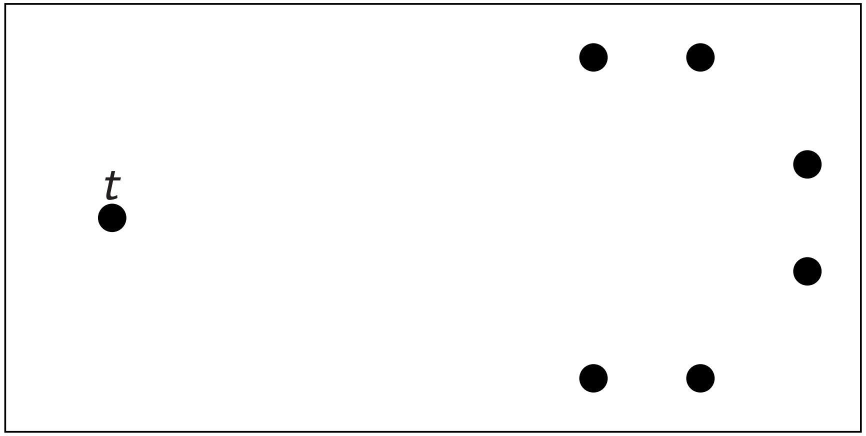
Polynomial for $|T| = 2$, but NP-hard for any fixed $|T| \geq 3$ [Dalhaus et al. 1994].

Trivial to solve in polynomial time for fixed k (in time $n^{O(k)}$).

Theorem: MULTIWAY CUT can be solved in time $4^k \cdot n^{O(1)}$, i.e., it is fixed-parameter tractable (FPT) parameterized by k.

MULTIWAY CUT

Intuition: Consider a $t \in T$. A subset of the solution S is a $(t, T \setminus t)$-separator.



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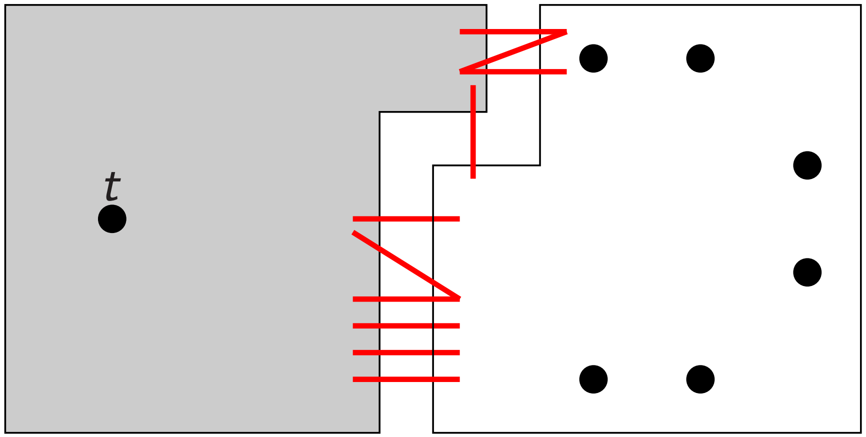

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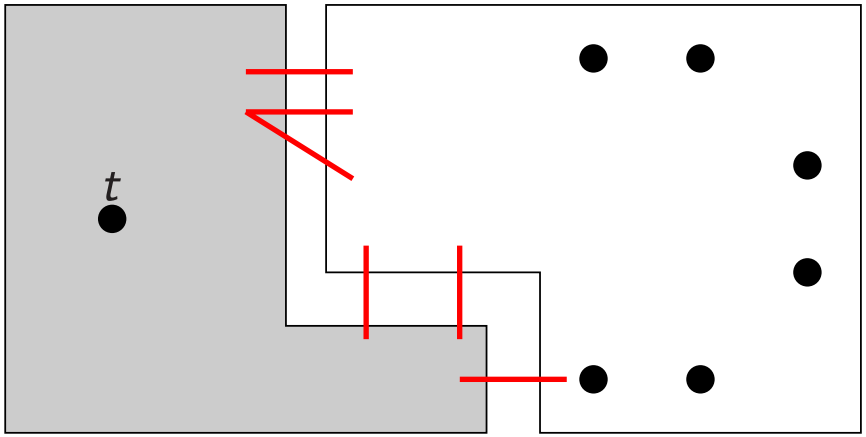

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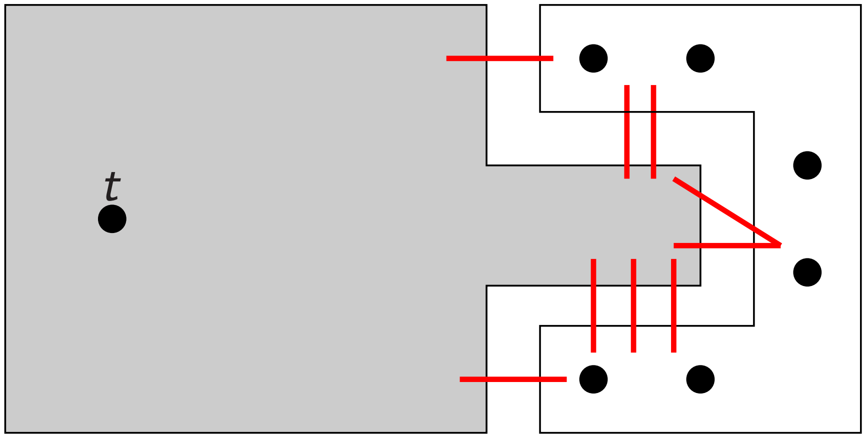

There are many such separators.

But a separator farther from t and closer to $T \setminus t$ seems to be more useful.

MULTIWAY CUT *and important separators*

Lemma: Let $t \in T$. The MULTIWAY CUT problem has a solution S that contains an important $(t, T \setminus t)$-separator.

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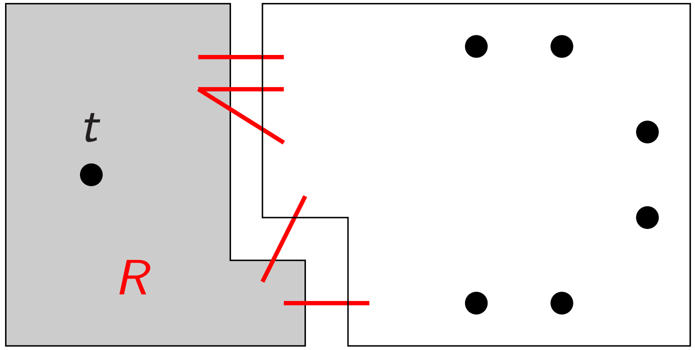

Multiway Cut *and important separators*

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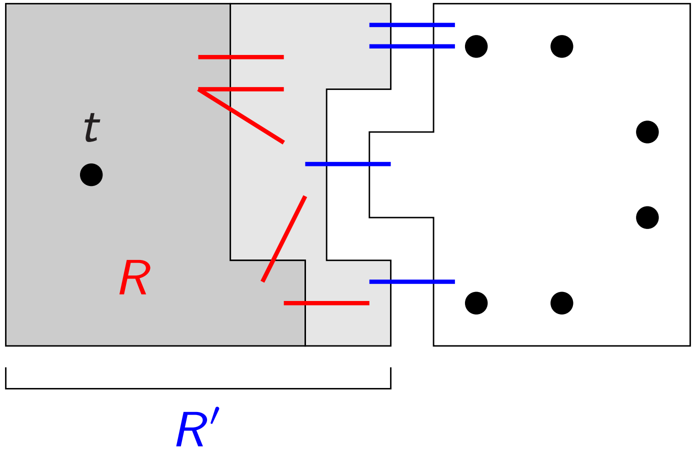

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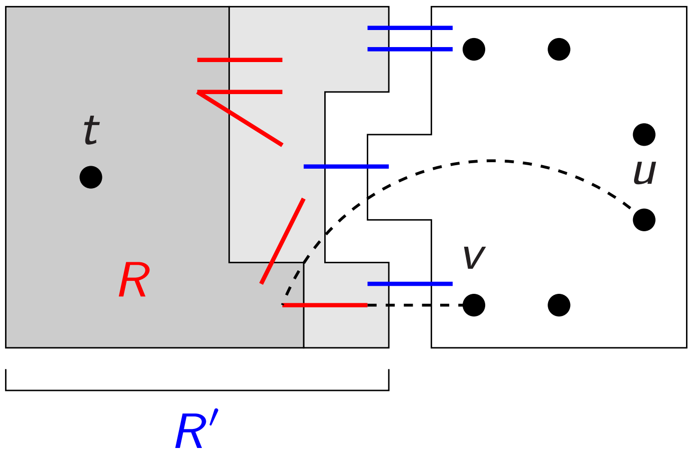

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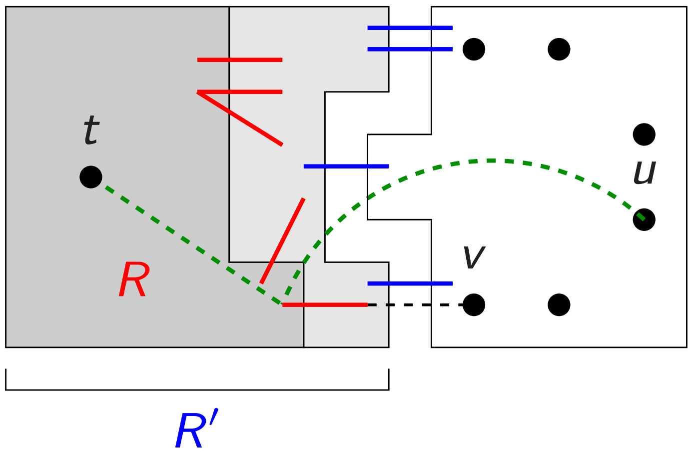

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Algorithm for MULTIWAY CUT

1. If every vertex of T is in a different component, then we are done.
2. Let $t \in T$ be a vertex with that is not separated from every $T \setminus t$.
3. Branch on a choice of an important $(t, T \setminus t)$ separator S of size at most k.
4. Set $G := G \setminus S$ and $k := k - |S|$.
5. Go to step 1.

We branch into at most 4^k directions at most k times.

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Better estimate of the search tree size:

- When choosing the important separator, $2k - \lambda$ decreases at each branching, until λ reaches 0.
- When choosing the next vertex t, λ changes from 0 to positive, thus $2k - \lambda$ does not increase.

Size of the search tree is at most 2^{2k}.

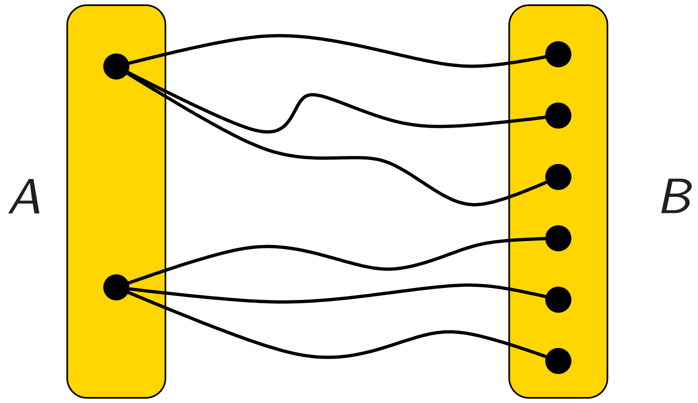
Open questions

- **Open:** Is there an $f(k) \cdot n^{O(1)}$ time algorithm for MULTIWAY CUT in directed graphs? Open even for $|T| = 2$.
- MULTITERMINAL CUT: pairs $(s_1, t_1), \ldots, (s_\ell, t_\ell)$ have to be separated by deleting k edges (vertices).
- MULTITERMINAL CUT can be solved in time $f(k, \ell) \cdot n^{O(1)}$.
- **Open:** Is there an $f(k) \cdot n^{O(1)}$ time algorithm for MULTITERMINAL CUT?

Spiders

Let A and B be two disjoint sets of vertices in G. A d**-spider** with center v is a set of d edge disjoint paths connecting $v \in A$ with B.

Suppose for simplicity that every vertex of A has degree d.



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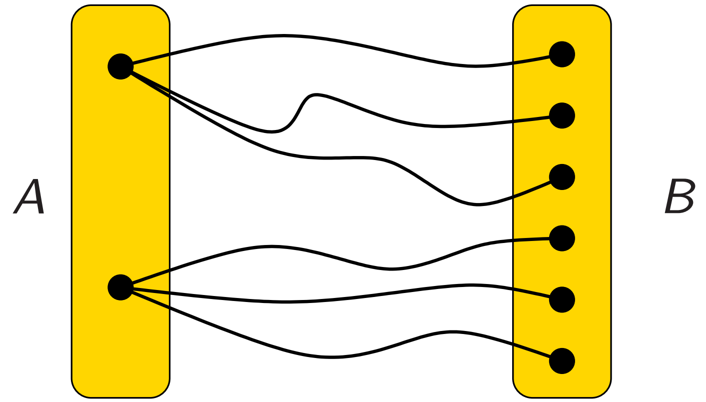

Theorem: There is a function $f(k, d) = 2^{O(kd)}$ such that for every graph G and disjoint sets A, B either

- there are k edge-disjoint d-spiders, or
- there is a set D of at most $f(k, d)$ edges that intersects every d-spider.

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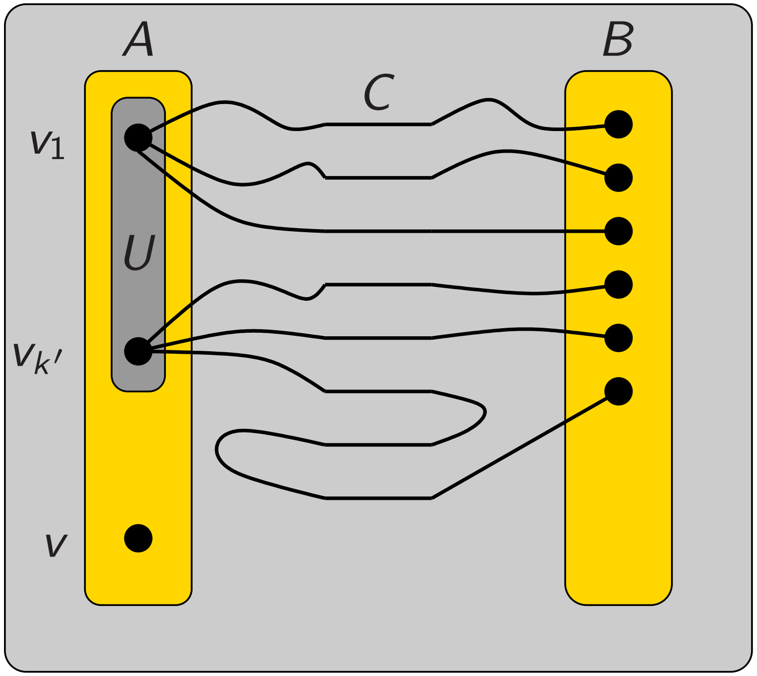
Proof: Suppose that there are $k' < k$ disjoint d-spiders with centers $U = \{v_1, \dots, v_{k'}\}$, but there are no $k' + 1$ disjoint spiders.

Let D be the union of all the important (v_i, B)-separators of size at most kd for $1 \le i \le k'$.
$\Rightarrow$ size of D is at most $f(k, d) := k \cdot 4^{kd} \cdot kd$.

We claim that D intersects every d-spider.

Spiders

Remember: D contains every important (v_i, B)-separator of size $\leq kd$.



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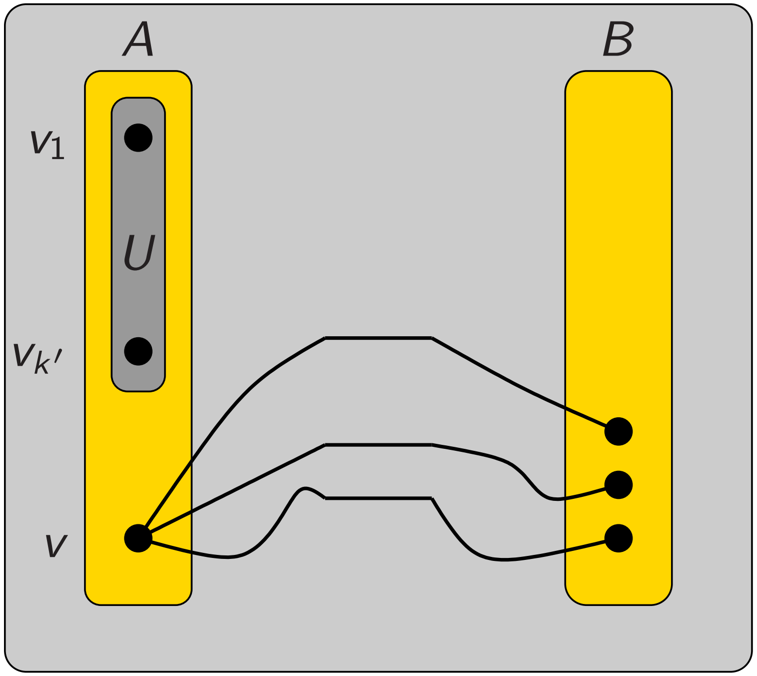
Consider a spider S with center v. As there are no $k' + 1$ spiders with centers $U \cup v$, there is a $(U \cup v, B)$-separator C with $|C| < (k' + 1)d$.



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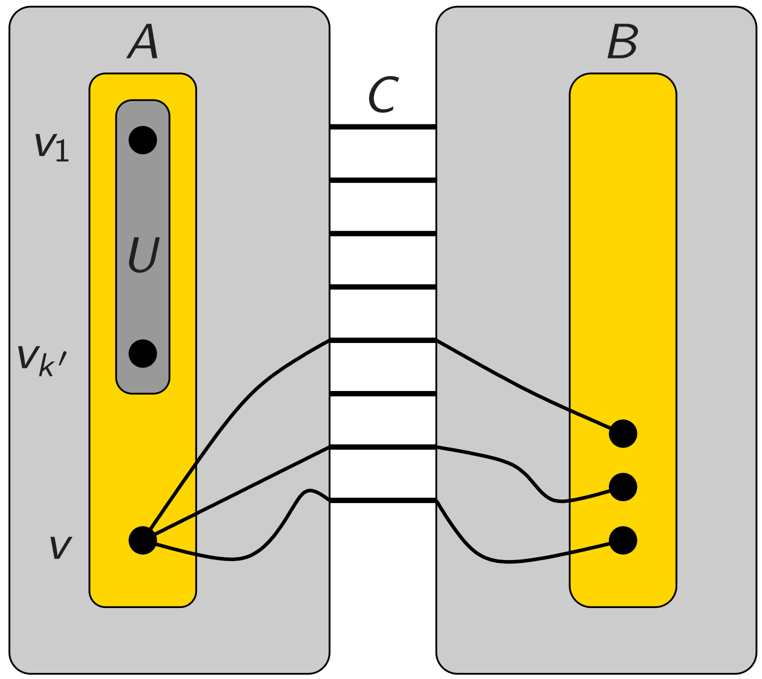

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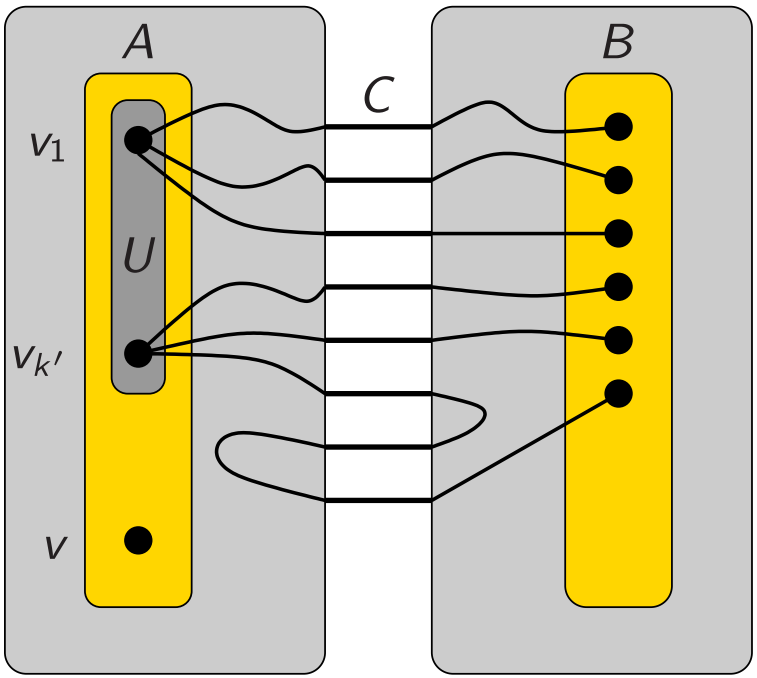

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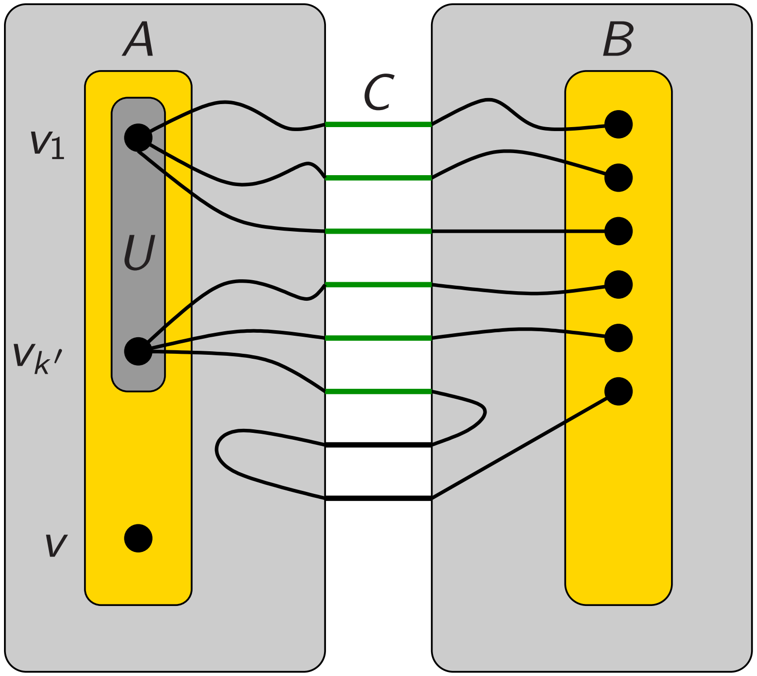
An edge of C is green if it is the first edge in C of any of the paths of the k' spiders

$\Rightarrow$ there are $k'd$ green edges.

$\Rightarrow$ there are $\leq d-1$ non-green edges.



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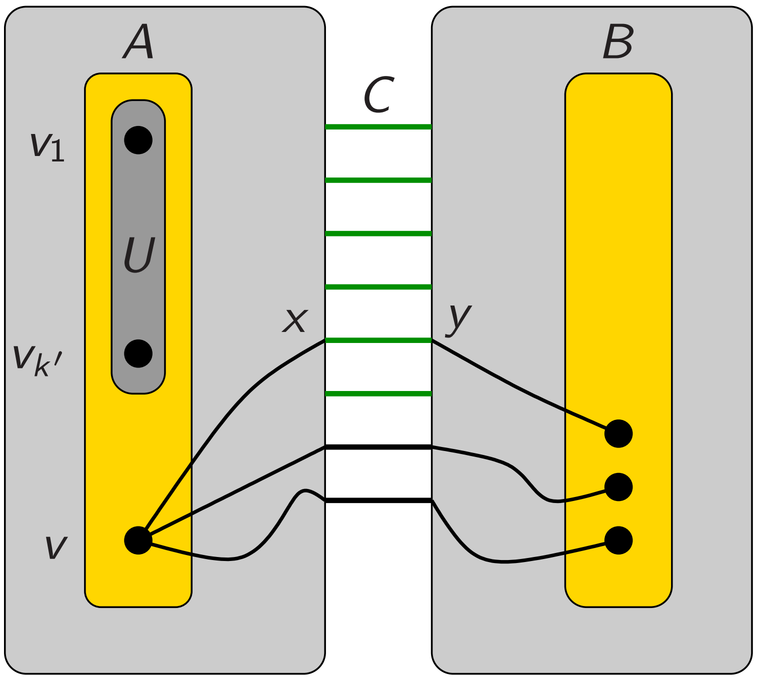
⇒ Spider S contains a green edge xy

⇒ Spider S connects x and B.



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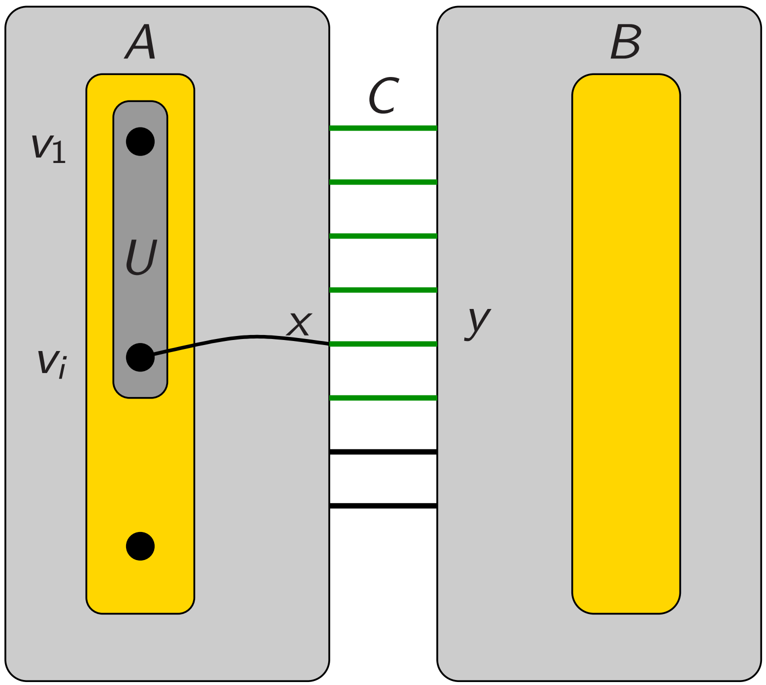

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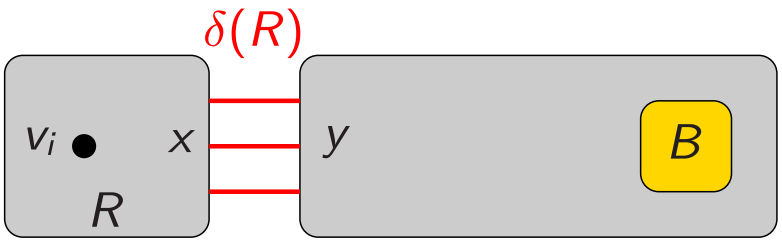

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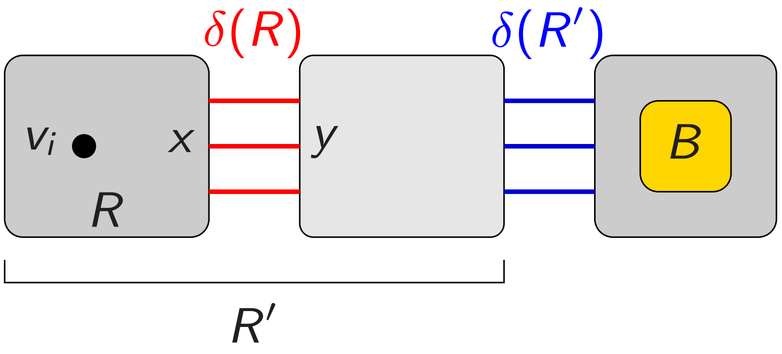
$\delta(R)$ is a (v_i, B)-separator of size $< kd$
$\Rightarrow$ D contains a separator $\delta(R')$ with $R \subseteq R'$.



$x \in R' \Rightarrow \delta(R')$ separates x and B
$\Rightarrow D \supseteq \delta(R')$ intersects the spider S.

Algorithmic questions

Packing

Theorem: [M. 2006] It can be decided in time $f(k,d) \cdot n^{O(1)}$ if there are k disjoint d-spiders.

Algorithm uses the following two ideas:

- A matroid describes which subset of edges incident to A can be the start edges of disjoint paths to B (well-known).
- Given a represented matroid whose elements are partitioned into blocks of size d, it can be decided in time $f(k,d) \cdot n^{O(1)}$ if there are k blocks whose union is independent [M. 2006].

More combinatorial algorithm?

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More combinatorial algorithm?

Covering

Can we find in $f(k,d) \cdot n^{O(1)}$ time k edges covering the d-spiders?

Conclusions

- A simple (but essentially tight) bound on the number of important separators.
- Useful for FPT algorithms.
- Erdős-Pósa property for spiders. Is the function $f(k, d)$ really exponential?
- Some open algorithmic questions.