ITC2+ CGT1

Repeat Midterm Test Grading Guide

General Rules. Disclaimer: Google translate has been used for this section. The purpose of the scoring guide is to ensure that the dissertations are evaluated uni- formly by the correctors. Therefore, the guide the main ideas for solving each task (at least one possible) and the marks assigned to them communicates sub-scores. The guide is not intended to detail the complete solution of the tasks description; the steps described can be considered as a sketch of a solution with a maximum score. The sub-scores indicated in the guide only accrue to the solver if the related idea is included in the dissertation as a step towards a clear, clearly described and justified solution. Thus, for example, stating the definitions and items in the material without knowing how to apply them does not deserve any points (even if any of the facts described are indeed used in the solution). Deciding the score based on the points indicated in the guide in light of the above is under the grader's full remedial authority. A partial score is awarded for each idea or partial solution from which, with a suitable addition, a flawless solution to the problem would have been obtained. If a solver starts several several substantially different solutions for a task, he can be assigned to at most one score. If all the solutions or parts of solutions described are correct or correct, then the solution initiative worth the most subpoints is evaluated. However, if amongst several solution attempts there is a correct solution but also an incorrect one (with a substantial error), and it is not clear from the dissertation which the solver considered as correct, then the solution with fewer points is evaluated (even if this score is 0). The sub-scores in the guide can be further divided if necessary. A good solution other than that described in the guide is, of course, worth a maximum point. Theorems can be stated without proof, but only those discussed in class.

1. Let the vertices of the graph G be all the 5-element subsets of a 12-element set, and two vertices be adjacent if and only if the corresponding subsets are disjoint. How many edges does G have?

   Solution:
   $|V(G)| = \binom{12}{5}$. (2 points)
   $\deg(v) = \binom{7}{5}, \forall v \in V(G)$, (2 points)
   because the neighbors of v are subsets of the complement of the set corresponting to v. (3 points)
   So $|E(G)| = \binom{12}{5} \cdot \binom{7}{5}/2 = 8316$, (1 points)
   by the degree-sum formula. (2 point)

2. Let G be the complete graph on the vertex set $V(G) = \{1, 2, \ldots, 10\}$. For every $1 \le i < j \le 10$ let the weight of the edge $\{i, j\}$ be $\lfloor \frac{2j-i}{3} \rfloor$ (where $\lfloor\,\rfloor$ denotes the lower integer part). Determine a minimum weight spanning tree in G.

   Solution:
   We'll use Kruskal's algorithm. (2 points)
   First we select 3 edges of weight 1 (1 point)
   the fourth one cannot be chosen, because... (1 point)
   then we select 3 edges of weight 2 (1 point)
   more edges of weight 2 cannot be selected, because... (1+1 points)
   Finally we select 3 edges of weight 3. (1 point)
   This way we selected $9 = |V(G)| - 1$ edges, and the algorithm stops. (1+1 points)
   The spanning tree can be (is) a Hamilton path. (0 points)

3. How many non-isomorphic complete bipartite graphs are there on 100 vertices which contain a Hamilton cycle? (In a complete bipartite graph all the verttices in one class are connected to all the vertices in the other class.)

Solution:

The complete bipartite graph is $K_{a,b}$ with $a \le b, a + b = 100$. (0 points)
If $a < b$, then if we delete the smaller class, we get b components, (2 points)
all of them are isolated vertices, (1 point)
so by the necessary condition there is no Hamilton cycle. (2 points)
But if $a = b = 50$, then there is a Hamilton cycle. (1 point)
either by describing it explicitely or by Dirac's theorem. (3 points)
So there is only one such graph: $K_{50,50}$. (1 point)

4. Let the two vertex classes of the bipartite graph $G(A, B; E)$ be $A = \{a_1, a_2, \dots, a_7\}$ and $B = \{b_1, b_2, \dots, b_7\}$. For each $1 \le i \le 7$ and $1 \le j \le 7$ let a_i and b_j be adjacent if the entry in the ith row and jth column of the matrix below is 1. Determine a maximum matching and a minimum covering set in G.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Solution:

There is a matching of size 6 (1's in different rows and columns). (2 points)
There are 6 covering vertices (the corresponding rows and columns contain all the 1's).. (3 points)
These are maximum and minimum:
We know that $\nu(G) \le \tau(G)$ holds for each graph. (1 point)
From the matching we have $\nu(G) \ge 6$, (1 point)
and from the covering set of vertices $\tau \le 6$. (1 point)
So $6 \le \nu(G) \le \tau(G) \le 6$, (1 point)
and $\nu(G) = 6$ and our matching is maximum, and $\tau = 6$ and our covering set of vertices is minimum. (1 point)

5. Let the graph G be the complement of a path on 6 vertices (with 5 edges). Determine $\chi_e(G)$, the edge-chromatic number of the graph G.

Solution:

The picture of the graph is ... (1 point)
$\Delta(G) = 4$. (1 point)

We know: $\chi_e(G) \geq \Delta(G) = 4$ (2 points)
There is a good coloring of the edges with 4 colors. (4 points)
So $\chi_e(G)$ is at most 4, and then it must be 4. (2 points)

6. * Let the network (G, s, t, c) be given, furthermore an edge e in G for which $c(e) > 0$ holds. Determine whether the statements below are true or not:
a) If there is a minimum s, t-cut C for which e goes out of X, then $f(e) = c(e)$ holds for all maximum flows f.
b) If $f(e) = c(e)$ holds for all maximum flows f, then there is a minimum s, t-cut C for which e goes out of X.

Solution:
a) TRUE. If f is a max. flow and C a min. cut, then in the proof of Proposition 3 we have equalities everywhere, so $f(e) = c(e)$. (3 points)
b) TRUE. By contradiction, if e does not go out of X for any min. cut, (1 point)
then $\min\{c(C) : e \text{ goes out of } X\} - c(\text{min.cut}) = \delta > 0$. (1 point)
Then we can decrease the capacity of e: let $c'(e) = \max\{c(e) - \delta, 0\} \geq 0$ (1 point)
In the new network the capacity of the min. cut is still the same (1 point)
so the value of the max. flow f is still the same, (1 point)
but $f(e) \leq c'(e) < c(e)$, (1 point)
a contradiction. (1 point)