Exercise-set 4.

1. Do the following graphs contain a Hamilton cycle? And a Hamilton path?

2. Let the vertices of the graph $G$ be the squares of a $5 \times 5$ chessboard, and two vertices be adjacent if and only if the corresponding squares have a common edge. The graph $G_1$ is obtained from $G$ by deleting a vertex corresponding to one of the corners of the chessboard from it (so $G_1$ has 24 vertices). The graph $G_2$ is obtained from $G$ by deleting two vertices corresponding to opposite corners of the chessboard from it (so $G_2$ has 23 vertices).
   a) Does $G_1$ contain a Hamilton cycle? And a Hamilton path?
   b) Does $G_2$ contain a Hamilton cycle? And a Hamilton path?

3. Let the vertex set of the graph $G$ be $V(G) = \{1, 2, \ldots, 20\}$. Let the vertices $x, y \in V(G)$ be adjacent in $G$ if $x \neq y$ and $x \cdot y$ is divisible by 3 or 5 (or both).
   a) Does $G$ contain a Hamilton path?
   b) Does $G$ contain a Hamilton cycle?

4. At least how many edges must be added to the graphs below so that the graphs obtained contain a Hamilton cycle?

5. The graph $G$ is a star on 101 vertices (i.e. $G$ has one vertex of degree 100 and hundred vertices of degree 1). At least how many edges must be added to $G$ so that the graph obtained contains a Hamilton cycle?

6. a) Show that it is impossible to visit each square of a $4 \times 4$ chessboard (exactly once) with a horse.
   b) Show that it is impossible to visit each square of a $5 \times 5$ chessboard (exactly once) with a horse such that in the 25th move we arrive back to the starting square.
   c) * (MT’19) Can we visit each square of a $3 \times 5$ chessboard exactly once with a horse?

7. (MT’21) The vertex set of a graph on 100 vertices can be divided into two classes and all the vertices in one class are connected to all the vertices in the other class. How many non-isomorphic such a graphs are there which contain a Hamilton cycle?

8. In a company of 12 everybody knows at least 6 others (acquaintances are mutual). Show that this company can be seated around a round table in such a way that everybody knows his/her neighbors.

9. a) (MT’03) The simple graph $G$ has 101 vertices. One of its vertices has degree 50, and all the other vertices have degree at least 51. Prove that $G$ contains a Hamilton cycle.
   b) The simple graph $G$ has 101 vertices. Two of its vertices have degree 50, and all the other vertices have degree at least 51. Prove that $G$ contains a Hamilton path.

10. In a company of 20 everybody knows the same number of people (acquaintances are mutual). Show that this company can be seated around a round table in such a way that either everybody knows his/her neighbors or nobody knows his or her neighbors.

11. * There are 50 guests at a banquet, each of them knows at least 5 people from the others. (Acquaintances are mutual.) No matter how we choose 3 or 4 from the guests they cannot sit down to a round table in such a way that everybody knows both of his/her neighbors. Show that in this case all the guests can be seated around a round table for 50 persons in such a way that any two people who sit next to each other, but don’t know each other have a common friend among the guests.

12. * In the simple graph $G$ on $2k + 1$ vertices each vertex has degree at least $k$. Prove that $G$ contains a Hamilton path.
13. * In a simple graph on 20 vertices the degree of each vertex is at least 9. Prove that we can add one new edge to the graph in such a way that the resulting graph contains a Hamilton path.

14. * In the simple graph $G$ on 201 vertices the degree of each vertex, except for $v$, is at least 101. About $v$ we only know that it is not an isolated vertex. Show that $G$ contains a Hamilton path.

15. * Show that if $G$ is a simple 9-regular graph on 16 vertices, then we can delete 8 edges of $G$ in such a way that the remaining graph contains an Euler circuit.

16. * Let $G$ be a simple graph on $2k$ vertices in which the degree of each vertex is $k - 1$, where $k > 1$ is an integer. Prove that we can add $k$ new edges to $G$ in such a way that the resulting graph contains a Hamilton cycle.

17. Determine whether the first two graphs below are bipartite or not:

18. (MT'09) At least how many edges must be deleted from the third graph above to get a bipartite graph?

19. (MT'10) 7 knights are put on a chessboard in such a way that each of them attacks at least two others. Show that there is such a knight among them which attacks three others.

20. (MT'16) Let the vertices of the graph $G$ be the all the 0-1 sequences of length 5, and two sequences be adjacent if they differ in exactly one position. Is $G$ a bipartite graph?

21. (MT++'16) Is there a simple bipartite graph on at least 5 vertices whose complement is also a bipartite graph?

22. (MT'17) In a graph on 99 vertices two vertices have degree 3, and the degree of the other vertices is 4. Show that the graph contains an odd cycle.

23. * Determine all the nonisomorphic simple graphs $G$ on 8 vertices for which $\chi(G) = 2$ but if we add any edge to $G$ (between two nonadjacent vertices) then for the graph $G'$ obtained this way $\chi(G') = 3$ holds.

24. * (MT'+03) Determine all the nonisomorphic simple graphs $G$ on $n$ vertices for which $\chi(G) = 3$ but if we delete any vertex from $G$ (together with the edges adjacent to it) then for the graph $G'$ obtained $\chi(G') = 2$ holds.

25. Determine the chromatic number of the graph of the regular octahedron. (The octahedron has 6 vertices and 8 triangular faces.)

26. Let the vertices of the graph $G$ be the squares of the chessboard, and two vertices be adjacent if and only if the corresponding squares can be reached from each other by one move of a rook. Determine $\chi(G)$, the chromatic number of $G$. (A rook in chess can move either horizontally or vertically, and in one move it can go to any square along the selected line.)

27. Let the vertices of the graph $G$ be the integers 1,2,...,100, and two vertices, $m$ and $n$ be adjacent if and only if $m + n$ is odd. Determine $\chi(G)$, the chromatic number of $G$.

28. (MT'+18) We add two non-adjacent edges to the complete bipartite graph $K_{3,3}$ in such a way that the resulting graph $G$ is simple. Determine $\chi(G)$, the chromatic number of $G$.

29. (MT'05, MT'+15) Determine the chromatic number of the graphs below:

30. (MT'14) Let $G$ be the graph obtained from a regular 8-sided polygon by adding all the shortest diagonals to it (i.e. $G$ has 8 vertices and 16 edges). Determine $\chi(G)$ and $\omega(G)$. 