Introduction to the Theory of Computing I.
Second Midterm Test
2019. December 6.

1. The vectors $\underline{x}, \underline{y}, \underline{z}, \underline{w}$ form a basis of $\mathbf{R}^4$. Determine the dimension of the subspace generated by the vectors $\underline{x} - \underline{y},\ \underline{z} - \underline{w},\ \underline{x} + \underline{y} + \underline{z} + \underline{w}$.

2. Determine for which values of the parameter p the system of equations below is consistent. If it has solutions, then determine all of them.
$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 &= 6 \\ 2x_1 + 6x_2 + (p+2)x_3 + (p+3)x_4 &= p+6 \\ 3x_1 + 7x_2 + 2x_3 &= 8 \end{aligned}$$

3. Evaluate the determinant below *using the original definition.* (So don't use any properties of the determinant, or theorems about it during the solution, but determine the value using the definition only.)
$$\begin{vmatrix} 0 & 2 & 8 & 7 & 1 \\ 0 & 4 & 2 & 3 & 0 \\ 3 & 6 & 3 & 3 & 9 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 5 & 0 & 2 & 0 \end{vmatrix}$$

4. For the matrix A below it holds that $A^3 = I$ (where I is the 3×3 identity matrix).
a) Evaluate the determinant of A.
b) Determine the bottom right entry of the inverse of A.
$$A = \begin{pmatrix} 9 & 13 & 13 \\ 1 & 3 & 2 \\ -8 & -13 & -12 \end{pmatrix}$$

5. Determine the rank of the matrix below depending on the parameter x.
$$\begin{pmatrix} x & x & 1 & x & x \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & x & 1 & 1 \end{pmatrix}$$

6. * Let V be a subspace of $\mathbf{R}^4$, C a 4×3, and A a 3×3 matrix. Moreover suppose that every column of the matrix $C \cdot A$ is in V, but the matrix C has a column (at least one) which is not in V. Prove that in this case $\det A = 0$.

The full solution of each problem is worth 10 points. Show all your work! Results without proper justification or work shown deserve no credit.
Notes and calculators (and similar devices) cannot be used.
The question denoted by an * is supposed to be more difficult.

(Approximate) grading: 0-23 points: 1, 24-32 points: 2, 33-41 points: 3, 42-50 points: 4, 50-60 points: 5.