ITC2 Second Repeat Midterm, June 3rd, 2020.

General Rules. Disclaimer: Google translate has been used for this section. The solutions have been translated by Padmini Mukkamala.

The purpose of the scoring guide is to ensure that the dissertations are evaluated uniformly by the correctors. Therefore, the guide the main ideas for solving each task (at least one possible) and the marks assigned to them communicates sub-scores. The guide is not intended to detail the complete solution of the tasks description; the steps described can be considered as a sketch of a solution with a maximum score.

The sub-scores indicated in the guide only accrue to the solver if the related idea is included in the dissertation as a step towards a clear, clearly described and justified solution. Thus, for example, stating the definitions and items in the material without knowing how to apply them does not deserve any points (even if any of the facts described are indeed used in the solution). Deciding the score based on the points indicated in the guide in light of the above is under the grader's full remedial authority.

A partial score is awarded for each idea or partial solution from which, with a suitable addition, a flawless solution to the problem would have been obtained. If a solver starts several several substantially different solutions for a task, he can be assigned to at most one score. If all the solutions or parts of solutions described are correct or correct, then the solution initiative worth the most subpoints is evaluated. However, if amongst several solution attempts there is a correct solution but also an incorrect one (with a substantial error), and it is not clear from the dissertation which the solver considered as correct, then the solution with fewer points is evaluated (even if this score is 0).

The sub-scores in the guide can be further divided if necessary. A good solution other than that described in the guide is, of course, worth a maximum point. Theorems can be stated without proof, but only those discussed in class.

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1. First solution: We will pick a letter at a time in order. (1 pont)
The first letter can be any of the 52 letters (small and capitals taken into consideration). (1 pont)
The second letter has only 50 options, because both the small and capital variation of the first letter cannot be used anymore. (1+1 pont)
Using similar reasoning, there are 48 options for the third letter, and proceeding thus, there will be 38 options for the last (eighth) letter. (1+1 pont)
So the number of passwords is $52 \cdot 50 \cdot 48 \cdot 46 \cdot 44 \cdot 42 \cdot 40 \cdot 38$, (2 pont)
and we multiply here because any one letter password can be continued in 50 different ways, and any two letter password can be continued in 48 ways, and so on. (2 pont)

Second solution: We first select the 8 letters that show up on the password, and since every letter can appear at most once, (1 pont)
this is precisely, $\binom{26}{8}$. (1 pont)
There are two options for each letter, it canbe either small or capital,

(1 pont)

so each such decision will double the number of passwords, so taking into small and capital letters into consideration, the number of ways of picking letters is $\binom{26}{8} \cdot 2^8$. (2 pont)

We also need to consider the ordering of these letters to get the total number of passwords. (1 pont)

Once the letters have been picked, they can be ordered in 8! ways. (1 pont)

So the final result, or the number of passwords is $\binom{26}{8} \cdot 2^8 \cdot 8!$, (1 pont)

which written in the form asked is $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 2^8$. (The 2^8 not written as multiplication can be ignored.) (2 pont)

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2. Let the number of vertices in G be n. Then any spanning tree of G has $n-1$ edges. (1 pont)

And since H has n vertices, its spanning tree also has $n-1$ edges. (1 pont)

Since G is simple, (1 pont)

and planar, it can have at most $3n-6$ edges. (1 pont)

So the number of edges remaining after the removal of the two spanning trees is at most $3n-6-2(n-1) = n-4$. (3 pont)

So the graph will have at least 4 components, (1 pont)

otherwise, we could connect it with 2 or fewer new edges, (1 pont)

which is a contradiction, since then it would be connect with at most $n-2$ edges. (1 pont)

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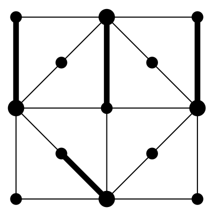
3. We can find a matching of 4 edges as shown in the figure below. (2 pont)



We will show that this is the maximum. We know that any matching has at most as many edges as the number of vertices in any vertex cover. (1 pont)

So we can show a vertex cover of 4 vertices, say S, then that will prove that the above matching is maximal. (2 pont)

because there cannot be a matching with 5 edges, since atleast one endpoint of each edge in the matching is in S.

(1 pont)

In the above picture, the vertices denoted by larger circles form a vertex cover of exactly 4 vertices, (2 pont)

we can see that every edge of the graph is incident to one of these vertices. This proves the claim. (2 pont)

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4. One such graph is the complete graph on 8 vertices. A solution which mentions this and has some reasoning written down for it, should get 2 points. No points should be awarded for simply writing K_8. 2 points are for seeing that chromatic number of K_8 is 8. 3 further points are for showing a coloring with 4 colors of the graph obtained after deleting a Hamiltonian cycle. The last 3 points are for showing that fewer than 4 colors are insufficient.

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5. First solution: We will pick vertices one at a time and show that the process can be continued until we pick 7 vertices in our independent set. The choice of the first vertex is arbitrary. (1 pont)
Remove all the neighbors of the first vertex from the graph. (1 pont)
We are left with a graph on at least 42 vertices. (1 pont)
since the maximum degree is at most 7. (1 pont)
From these, pick the second vertex and delete this and its neighbors from the graph, (1 pont)
and repeat this process while there are vertices left in the graph. (1 pont)
Since every step deletes at most 8 vertices from the graph, (1 pont)
(since the maximum degree cannot be greater the subgraph obtained after deleting vertices), (1 pont)
after the sixth iteration, the graph will have at least 2 vertices in it, picking one of which, we get an independent set of size 7. (1 pont)
We further note that for every vertex picked, it follows that it is not connected to any vertex picked in the previous iterations (since neighbors were deleted), hence the selected vertices do indeed form an independent set. (1 pont)

Second solution: We know that vertices of every graph G can be properly colored with $\Delta(G)+1$ colors using the greedy coloring algorithm, so the given graph can be colored with 8 colors. (3 pont)
Every color class is an independent set. (3 pont)
We claim that there must be a color class with at least 7 vertices, (3 pont)
because if not, then the total vertices in all color classes will be at most $8 \cdot 6 = 48$, which is less than the number of vertices in the graph. (1 pont)

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6. One such graph G (for example), is the graph obtained from K_5 by deleting an edge. (1 pont)
To see the correctness of this example, we first show that its edge chromatic number is 5. Notice that G has 9 edges, (1 pont)
while a maximum matching in G can have at most 2 edges, (1 pont)
since a matching of 3 edges would be possible only if the number of vertices in the graph

was at least 6. (1 pont)
Since the color classes in any proper edge coloring constitute matchings, (1 pont)
at least 5 colors are needed to color 9 edges. (1 pont)
On the other hand, we can see that 5 colors are sufficient to color the edges of G, either by showing a coloring, or by Vizing's theorem. (1 pont)
On removing the edges of a Hamiltonian cycle from G, we obtain a path of 4 edges, (1 pont)
whose edges can be colored by alternating colors, so its edge chromatic number is just 2, (1 pont)
while one color is not sufficient because of vertices of degree 2. (1 pont)