Exercise-set 9.
Solutions

1. $\Delta(G) = 4 \implies \chi_e(G) \geq 4$, and the edges of G can be colored with 4 colors $\implies \chi_e(G) \leq 4$.
2. $\chi_e(K_5) \geq e/\nu = 10/2 = 5$ and $\chi_e(K_5) \leq \Delta(K_5) + 1 = 5$, so $\chi_e(K_5) = 5$.
   $\chi_e(K_6) \geq \chi_e(K_5) = 5$, and the edges of K_6 can be colored with 5 colors $\implies \chi_e(K_6) \leq 5$.
   (In general, $\chi_e(K_{2n+1}) = 2n + 1$ and $\chi_e(K_{2n}) = 2n - 1$.)
3. $\chi_e(K_{20}) = 19$ (ex. 3.), and a round corresponds to edges of the same color.
4. $\chi_e(G) \geq \chi_e(K_5) = 5$, and and the edges of G can be colored with 5 colors $\implies \chi_e(G) \leq 5$.
5. $|E(G)| = 1999 \cdot 10/2 = 9995$, $\nu(G) \leq 1999/2 = 999 \implies \chi_e(G) \geq 9995/999 > 10$ and $\chi_e(G) \leq \Delta(G) + 1 = 11 \implies \chi_e(G) = 11$.
6. a) $\chi_e(G) \geq e/\nu = (2k \cdot 3 + 2)/2k > 3$ (since $|V(G)|$ is odd) and $\chi_e(G) \leq \Delta(G) + 1 = 4 \implies \chi_e(G) = 4$.
   b) Delete the cut-edge and use a).
7. a) $\chi_e(G) \geq e/\nu = 15/2 > 7$ and the edges of G can be colored with 8 colors $\implies \chi_e(G) = 8$.
   b) $\chi_e(G) \geq e/\nu = 15/2 > 7$ and the edges of G can be colored with 8 colors $\implies \chi_e(G) = 8$.
8. $\chi_e(G) \geq e/\nu = 71/7 > 10$ and $\chi_e(G) \leq \Delta(G) + 1 = 11 \implies \chi_e(G) = 11$.
9. Any color class of edges forms a perfect matching (covers all the vertices).
10. $\nu(G) \geq e/\chi_e \geq 16/5 > 3$ (since $\chi_e(G) \leq \Delta(G) + 1 = 5$), and $\nu(G) \leq 9/2$.
11. For a k-regular graph on 9 vertices $\chi_e(G) = k + 1$, and $\overline{G}$ is $8 - k$-regular $\implies \chi_e(\overline{G}) \geq 9 - k$.
12. a) The edges of G can be colored with 3 colors; 2 colors for the Hamilton cycle (since it has an even length, because $|E| = 3|V|/2$ is an integer), and one for the remaining edges.
    b) The edges of it cannot be colored with 3 colors.
    c) The edges of any 2 colors form a Hamilton cycle.
13. The edges of the original graph can be colored with $\chi_e(G') + 2$ colors.
14. Yes, $K_5 \setminus \{\text{one edge}\}$ is like that.
15. $G =$(rows, colums; selected squares) is a 3-regular bipartite graph $\implies \chi_e(G) = 3$.
16. G is bipartite $\implies \chi_e(G) = \Delta(G) = 6$; or give a concrete edge-coloring.
17. $G =$ two vertex-disjoint paths (which are bipartite) and a 5-regular bipartite graph $\implies \chi_e(G) = 2 + 5 = 7$.
18. $\nu(G) \geq e/\chi_e \geq 10/3 > 3$ (since $\chi_e(G) = \Delta(G) \leq 3$).
19. $G =$ two vertex-disjoint cycles (which are bipartite) and a bipartite graph with $\Delta(G) = 8 \implies \chi_e(G) = 2 + 8 = 10$.
20. G is bipartite $\implies \chi_e(G) = \Delta(G) = 5$; or give a concrete edge-coloring.
21. The cut with $X = \{S, C, D, F\}$ has capacity 15.
22. No. Either find the max flow (which is 20), or notice that the capacity of a cut cannot be 19 (all the capacities are divisible by 3 except for 5), and use the Ford-Fulkerson theorem.