Exercise-set 4.
Solutions

1. There are 36 minimum weight spanning trees of weight 19.
2. There are 99! minimum weight spanning trees of weight $2 + 3 + \cdots + 100 = 5049$.
3. The weight of a minimum weight spanning tree is 150.
4.
5. By Kruskal's algorithm: the other edges of C can be selected before e.
6. Not possible; possible.
7. a) $|V(G)| = \binom{8}{2} = 28$, $\deg(v) = \binom{6}{2} = 15\ \forall v \in V(G) \implies$ no Euler-circuit.
   b) $|V(G)| = \binom{6}{3} = 20$, $\deg(v) = 1+\binom{3}{2}\cdot\binom{3}{1} = 10\ \forall v \in V(G)$, and G is connected $\implies\ \exists$ Euler-circuit.
8. $|V(G)| = 2^4 = 16$, $\deg(v) = \binom{4}{2} = 6\ \forall v \in V(G)$, but G is not connected $\implies$ no Euler-circuit.
9. Construct a graph G: $V(G) =$ children, and u and v are adjacent $\iff$ not next to each other in the circle. This graph contains an Euler-circuit ($\deg(v) = 8\ \forall v \in V(G)$, connected). Most number of passes = length of an Euler-circuit = $|E(G)| = 44$.
10. Construct a graph G: $V(G) =$ digits $= \{0, 1, \ldots, 9\}$, and u and v are adjacent $\iff\ u+v \neq 9$. This graph contains an Euler-circuit ($\deg(v) = 8\ \forall v \in V(G)$, connected) $\iff\ \exists n$.
11. Construct a graph G: $V(G) =$ letters, and u and v are adjacent $\iff$ can stand next to each other. This graph contains an Euler-circuit ($\deg(v) = 30$ for vowels and $\deg(v) = 10$ for consonants, connected). Length of the longest sequence of letters = length of an Euler-circuit + 1 = $|E(G)|+1 = \binom{10}{2} + 10 \cdot 21 + 1 = 256$.
12. a) Add k new edges ($\implies \exists$ Euler-circuit), then delete them.
    b) No: each trail eliminates ≤ 2 odd degrees from G.
13. Equivalently: at least how many edges have to be deleted, s.t. the remaining graph contains an Euler-trail? At least 2 ($\exists$ 6 vertices of odd degree in G) $\implies$ length of a trail = $|E(G)| - 2 = 2 \cdot 4 + 5 \cdot 5 - 2 = 31$.
14. All the vertices have odd degrees, but at most 2 is possible for a trail $\implies$ need to delete at least 4 edges. This is enough, if we delete 4 independent edges.
15. a), b) There can be at most 2 components $\implies$ adding one edge can make it connected, and the degrees will be OK.
16. $r =$ 1,2,3,4,5,7,9 NO; $r =$ 6,8 YES (each degree is even + connected).
17. YES: both components contain 2 vertices of odd degree.
18. a) There is no Hamilton cycle: if we delete 4 vertices we get 5 components.
    There is a Hamilton path: draw.
    b) There is no Hamilton cycle: if we delete 2 vertices we get 3 components.
    There is a Hamilton path: draw.
    c) There is no Hamilton cycle: if we delete 4 vertices we get 5 components.
    There is a Hamilton path: draw.
    d) There is no Hamilton cycle: if we delete 12 vertices we get 13 components.
    There is a Hamilton path: draw.
19. a) Yes (draw); yes.
    b) No (delete 11 vertices); yes (draw).
20. a) No (delete the 9 vertices divisible by 3 or 5).
    b) No as well.
21. a) If we delete 2 vertices we get 3 components $\implies$ need to add at least 1 edge. That is enough (draw).
    b) If we delete 2 vertices we get 4 components $\implies$ need to add at least 2 edges. That is enough (draw).

22. If we delete 1 vertex (the center) we get 100 components $\Longrightarrow$ need to add at least 99 edges. That is enough (if we add a path).

23. a) Construct a graph G: $V(G)$ = squares, and the edges are the possible moves of the horse. This graph contains no Hamilton path: if we delete the 4 middle vertices we get 6 components.
b) The graph contains no Hamilton cycle: if we delete 12 vertices we get 13 components.
c) No: if we delete 5 vertices we get 7 components.

24. There is no Hamilton cycle: if we delete 7 vertices we get 8 components.

25. No: if we delete 3 vertices we get 4 components.