Exercise-set 9.

Solutions

1. \( \Delta(G) = 4 \implies \chi_e(G) \geq 4 \), and the edges of \( G \) can be colored with 4 colors \implies \chi_e(G) \leq 4.

2. Edges of the same color are independent.

3. \( \chi_e(K_5) \geq e/\nu = 10/2 = 5 \) and \( \chi_e(K_5) \leq \Delta(K_5) + 1 = 5 \), so \( \chi_e(K_5) = 5 \).
\( \chi_e(K_6) \geq \chi_e(K_5) = 5 \), and the edges of \( K_6 \) can be colored with 5 colors \implies \chi_e(K_6) \leq 5. 
  (In general, \( \chi_e(K_{2n+1}) = 2n + 1 \) and \( \chi_e(K_{2n}) = 2n - 1 \).)

4. \( \chi_e(K_{20}) = 19 \) (ex. 3.), and a round corresponds to edges of the same color.

5. \( \chi_e(G) \geq \chi_e(K_5) = 5 \), and the edges of \( G \) can be colored with 5 colors \implies \chi_e(G) \leq 5.

6. \( |E(G)| = 1999 \cdot 10/2 = 9995 \), \( \nu(G) \leq 1999/2 = 999 \implies \chi_e(G) \geq 9995/999 > 10 \) and \( \chi_e(G) \leq \Delta(G) + 1 = 11 \implies \chi_e(G) = 11. \)

7. a) \( \chi_e(G) \geq e/\nu = (2k \cdot 3 + 2)/2k > 3 \) (since \( |V(G)| \) is odd) and \( \chi_e(G) \leq \Delta(G) + 1 = 4 \implies \chi_e(G) = 4. \)

  b) Delete the cut-edge and use a).

8. a) \( \chi_e(G) \geq e/\nu = 15/2 > 7 \) and the edges of \( G \) can be colored with 8 colors \implies \chi_e(G) = 8.

  b) \( \chi_e(G) \geq e/\nu = 15/2 > 7 \) and the edges of \( G \) can be colored with 8 colors \implies \chi_e(G) = 8.

9. \( \chi_e(G) \geq e/\nu = 71/7 > 10 \) and \( \chi_e(G) \leq \Delta(G) + 1 = 11 \implies \chi_e(G) = 11. \)

10. Any color class of edges forms a perfect matching (covers all the vertices).

11. \( \nu(G) \geq e/\chi_e \geq 16/5 > 3 \) (since \( \chi_e(G) \leq \Delta(G) + 1 = 5 \), and \( \nu(G) \leq 9/2. \).

12. \( G \) = (rows, columns; selected squares) is a 3-regular bipartite graph \implies \chi_e(G) = 3.

13. \( G \) is bipartite \implies \chi_e(G) = \Delta(G) = 6 \); or give a concrete edge-coloring.

14. \( G \) = two vertex-disjoint paths (which are bipartite) and a 5-regular bipartite graph \implies \chi_e(G) = 2 + 5 = 7.

15. \( \nu(G) \geq e/\chi_e \geq 10/3 > 3 \) (since \( \chi_e(G) = \Delta(G) \leq 3 \).)

16. \( G \) = two vertex-disjoint cycles (which are bipartite) and a bipartite graph with \( \Delta(G) = 8 \implies \chi_e(G) = 2 + 8 = 10. \)

17. \( G \) is bipartite \implies \chi_e(G) = \Delta(G) = 5 \); or give a concrete edge-coloring.

18. For a \( k \)-regular graph on 9 vertices \( \chi_e(G) = k + 1 \), and \( \overline{G} \) is \( 8 - k \)-regular \implies \chi_e(\overline{G}) \geq 9 - k. \)

19. a) The edges of \( G \) can be colored with 3 colors; 2 colors for the Hamilton cycle (since it has an even length, because \( |E| = 3|V|/2 \) is an integer), and one for the remaining edges.

  b) The edges of it cannot be colored with 3 colors.

  c) The edges of any 2 colors form a Hamilton cycle.

20. The cut with \( X = \{S, C, D, F\} \) has capacity 15.

21. No. Either find the max flow (which is 20), or notice that the capacity of a cut cannot be 19 (all the capacities are divisible by 3 except for 5), and use the Ford-Fulkerson theorem.