Exercise-set 8.
Solutions

1. If k committees have at least k members together, for $k = 1, 2, \ldots$ (Hall's condition).

2. a) Yes.
   b) No (H, J, L, M like only B, E, F).

3. (a) Count the number of edges between A and B in two ways.
   (b) Count the number of edges between X and $N(X)$ in two ways.
   (c) Frobenius' theorem.

4. Count the number of edges between X and $N(X)$ in two ways.

5. Use Hall's condition for the (people,chocolates;liking) bipartite graph for $|X| \leq n$ and $|X| \geq n+1$, resp.

6. Use Hall's condition for $|X| \leq \frac{n}{2}$ and $|X| \geq \frac{n}{2}$, resp.

7. There is a non-connected counterexample.

8. Can select the edges greedily or use Hall's condition.

9. a) Use Frobenius' theorem.
   b) Use Hall's theorem or unite the vertices of degree 3 and use Ex. 3.

10. No perfect matching: $N(\{a_1, a_2, a_4, a_6, a_8\}) = \{b_2, b_3, b_6, b_8\}$.

11. A 2-regular bipartite graph is the union of vertex-disjoint even cycles.

12. Hall's condition holds for the (rows, colums; coins) bipartite graph.

13. Hall's condition holds for the (figures, sets; containment) bipartite 4-regular (multi)graph.

14. $\nu(G) = \tau(G) = 8$, a minimum covering set is $\{a_2, a_3, a_6, a_8, b_1, b_4, b_7, b_9\}$.

15. $\nu(G) = \tau(G) = 100$, $\rho(G) = 102$, a maximum matching e.g. is $\{\{a_i, b_{i+1}\},\ i = 1, 2, \ldots, 100\}$.

16. a) $\nu(G) = \tau(G) = 6$,
    b) $\nu(G) = \tau(G) = 6$,
    c) $\nu(G) = \tau(G) = 9$,
    d) $\nu(G) = \tau(G) = 6$.

17. a) The two endpoints of the edges in the matching can get the same color.
    b) $\overline{G}$ is a regular bipartite graph $\Longrightarrow$ $\omega(G) = 50$ and also $\overline{G}$ contains a perfect matching $\Longrightarrow$ $\chi(G) = 50$.