Combinatorics and Graph Theory 1.

Exercise-set 6.
Solutions

1. The first graph is not bipartite (contains 5-cycles), but the second graph is.
2. Deleting 2 edges are enough, but less is not, since $\exists$ 2 edge-disjoint odd cycles in G.
3. The graph determined by the knights and attacks is bipartite (the two classes are to the white and black squares), and each of its degrees is at least 2 $\Longrightarrow$ $\exists$ a degree ≥ 3.
4. Yes (the two classes of vertices are sequences with an even or odd number of 1's, resp.).
5. No (the complement contains a triangle).
6. The vertices cannot be divided into two classes (degrees).
7. Complete bipartite graphs are like that.
8. The graphs are exactly the odd cycles (so in particular n must be odd). G must contain an odd cycle (otherwise $\chi(G') = 2$), and cannot contain more vertices or edges.
9. $\omega(G) = 3 \implies \chi(G) \geq 3$, and G can be colored with 3 colors $\implies \chi(G) \leq 3$.
10. $\omega(G) = 8$ (each row and column is a clique) $\implies \chi(G) \geq 8$, and G can be colored with 8 colors (colors are diagonal) $\implies \chi(G) \leq 8$.
11. G is bipartite (the two classes of vertices are the even and odd numbers, resp.) $\implies \chi(G) = 2$
12. a), b) $\omega(G) = 3 \implies \chi(G) \geq 3$, but G cannot be colored with 3 colors (proof!) $\implies \chi(G) \geq 4$. G can be colored with 4 colors $\implies \chi(G) \leq 4$.
13. $\omega(G) = 3 \implies \chi(G) \geq 3$, but G cannot be colored with 3 colors (proof!) $\implies \chi(G) \geq 4$. G can be colored with 4 colors $\implies \chi(G) \leq 4$.
14. $\chi(G) \geq \lceil n/2 \rceil$ (at most 2 vertices can get the same color), and G can be colored with this many colors $\implies \chi(G) = \lceil n/2 \rceil$.
15. $\omega(G) = 10$ (any 10 consecutive numbers form a clique) $\implies \chi(G) \geq 10$, and G can be colored with 10 colors (periodically) $\implies \chi(G) \leq 10$.
16. $\omega(G) = 5$ ($\{1, 8, 15, 22, 29\}$ is a clique) $\implies \chi(G) \geq 5$, and G can be colored with 5 colors $\implies \chi(G) \leq 5$.
17. $\omega(G) = 11$ ($\{10, 11, \ldots, 20\}$ is a clique) $\implies \chi(G) \geq 11$, and G can be colored with 11 colors $\implies \chi(G) \leq 11$.
18. $\omega(G) = 4$ (the powers of 2 form a clique) $\implies \chi(G) \geq 4$, and G can be colored with 4 colors (using the same color between consecutive powers of 2) $\implies \chi(G) \leq 4$.
19. $\omega(G) = 11$ (prime numbers and 1 form a clique) $\implies \chi(G) \geq 11$, and G can be colored with 11 colors $\implies \chi(G) \leq 11$.
20. G is K_{10} with a perfect matching deleted. $\omega(G) = 5 \implies \chi(G) \geq 5$, and G can be colored with 5 colors $\implies \chi(G) \leq 5$.
21. $\omega(G) = 8 \implies \chi(G) \geq 8$, and G can be colored with 8 colors $\implies \chi(G) \leq 8$.
22. $\omega(G) = 6 \implies \chi(G) \geq 6$, and G can be colored with 6 colors $\implies \chi(G) \leq 6$.
23. $\omega(G) = 9 \implies \chi(G) \geq 9$, and G can be colored with 9 colors $\implies \chi(G) \leq 9$.
24. a) There must be at least one edge between any 2 color classes.
    b) Otherwise we could put all the vertices of a color class into other color classes.
25. A proper coloring of G can be given by pairs of colors from G_1 and G_2.
26. Use the greedy coloring in the original (increasing) order of the vertices.
27. Order the vertices: first the exceptional ones, then the rest, and use the greedy coloring.
28. Use the greedy coloring in the decreasing order of the degrees.
29. Keep choosing vertices of degree at most 5 (in the remaining graph), then use the greedy coloring in the opposite order.