# Exercise-set 5.
# Solutions

1. Not possible; possible.
2. $|V(G)| = \binom{8}{2} = 28$, $\deg(v) = \binom{6}{2} = 15\ \forall v \in V(G) \implies$ no Euler-circuit.
3. $|V(G)| = 2^4 = 16$, $\deg(v) = \binom{4}{2} = 6\ \forall v \in V(G)$, but $G$ is not connected $\implies$ no Euler-circuit.
4. $|V(G)| = 2^n$, $\deg(v) = n\ \forall v \in V(G)$, and $G$ is connected $\implies\ \exists$ Euler-circuit if and only if $n$ is even.
5. Construct a graph $G$: $V(G) =$ children, and $u$ and $v$ are adjacent $\iff$ not next to each other in the circle. This graph contains an Euler-circuit ($\deg(v) = 8\ \forall v \in V(G)$, connected). Most number of passes = length of an Euler-circuit = $|E(G)| = 40$.
6. Construct a graph $G$: $V(G) =$ digits $= \{0, 1, \ldots, 9\}$, and $u$ and $v$ are adjacent $\iff\ u+v \neq 9$. This graph contains an Euler-circuit ($\deg(v) = 8\ \forall v \in V(G)$, connected) $\iff\ \exists n$.
7. a) Add $k$ new edges ($\implies \exists$ Euler-circuit), then delete them.
   b) No: each trail eliminates $\leq 2$ odd degrees from $G$.
8. 4 vertices have odd degrees, and not all of them are adjacent $\implies$ 1 edge is enough.
9. Yes: connected, and each degree is even.
10. There can be at most 2 components $\implies$ adding one edge can make it connected, and the degrees will be OK.
11. A circuit passes from $S$ to the complement of $S$ and back an even number of times.
12. Yes (delete $e$ and $f$).
13. a) False (if we delete a cycle component).
    b) True: connected and the degrees are even.
    c) False, $G'$ is not necessarily connected.
14. a) There is no Hamilton cycle: if we delete 4 vertices we get 5 components.
    There is a Hamilton path: draw.
    b) There is no Hamilton cycle: if we delete 2 vertices we get 3 components.
    There is a Hamilton path: draw.
    c) There is no Hamilton cycle: if we delete 4 vertices we get 5 components.
    There is a Hamilton path: draw.
15. a) Yes (draw); yes.
    b) No (delete 11 vertices); yes (draw).
16. a) No (delete the 9 vertices divisible by 3 or 5).
    b) No as well.
17. Yes, we can construct it recursively from $n = 2$.
18. a) If we delete 2 vertices we get 3 components $\implies$ need to add at least 1 edge. That is enough (draw).
    b) If we delete 2 vertices we get 4 components $\implies$ need to add at least 2 edges. That is enough (draw).
19. If we delete 1 vertex (the center) we get 100 components $\implies$ need to add at least 99 edges. That is enough (path).
20. a) Construct a graph $G$: $V(G) =$ squares, and the edges are the possible moves of the horse. This graph contains no Hamilton path: if we delete the 4 middle vertices we get 6 components.
    b) Construct a graph $G$: $V(G) =$ squares, and the edges are the possible moves of the horse. This graph contains no Hamilton cycle: if we delete the 12 vertices we get 13 components.
21. By contradiction: if $C$ doesn't contain all the vertices of $G$, then we could get a longer cycle.

22. Construct a graph $G$: $V(G)$ = people, and the edges are the acquaintances. Then $\deg(v) \geq 6 = 12/2$ $\Longrightarrow$ by Dirac's theorem $\exists$ a Hamilton cycle.

23. The condition in Ore's theorem holds for $G \Longrightarrow \exists$ a Hamilton cycle.

24. Construct a graph $G$: $V(G)$ = people, and the edges are the acquaintances. $G$ is $k$-regular for some $k$. If $k \geq 10 \Longrightarrow G$ contains a Hamilton cycle, if $k \leq 9 \Longrightarrow \overline{G}$ contains a Hamilton cycle.

25. Construct a graph $G$: $V(G)$ = people, and the edges are the acquaintances. Then $G$ contains no cycles of length 3 or 4. We need to show that $\exists$ a Hamilton cycle in $G_1$, where $G_1$ is obtained from $G$ by adding edges between the second neighbors, i.e. and $u$ and $v$ are adjacent in $G_1 \Longleftrightarrow$ the 2 people know each other or they have a common friend. Then $\deg_{G_1}(v) \geq 5 + 5 \cdot 4 = 25$ (using the property of $G$) $\Longrightarrow$ by Dirac's theorem $\exists$ a Hamilton cycle in $G_1$.

26. a) A cycle on $n$ vertices is like that (check).
    b) E.g. $K_7$ with the edge $\{u, v\}$ missing and the 8th vertex is connected to $u$.

27. Add a new vertex to $G$, and connect it to all the old vertices. Then the new graph contains a Hamilton cycle from which we can get a Hamilton path of $G$.

28. Delete $v$ from $G$. Then the new graph contains a Hamilton cycle from which we can get a Hamilton path of $G$.

29. The 8 edges must have pairwise no common endpoints (i.e. be independent). Every second edge of a Hamilton cycle will do (which exists because $\deg(v) \geq n/2 \ \forall v$).

30. We can add the edges of a Hamilton cycle of $\overline{G}$.

31. Need to add $k$ pairwise non-adjacent edges (from $\overline{G}$). $\overline{G}$ contains a Hamilton cycle ($\deg_{\overline{G}}(v) = k, \ \forall v \in V(G)$). Every second edge of it will do.