Exercise-set 4.
Solutions

1. a) Yes (2 triangles),
   b) No ($n - e + r = 2$).
2. No ($n - e + 2n = 2 \implies e = 3n - 2$, contradiction).
3. $n = 8,\ r = 10$.
4. $n = 20,\ r = 12,\ k \cdot r = 2e,\ n - e + r = 2 \Longrightarrow k = 5$ (dodecahedron).
5. If there are a quadrangular and b octogonal faces, then $3n = 4a + 8b,\ 2e = 3n,\ r = a + b \Longrightarrow a - b = 6$.
6. Like the proof of Corollary 1 (of Euler's theorem), but with equalities.
7. a) Otherwise $3n \le e \le 3n - 6$, contradiction.
   b) I k vertices have degree 5 and $n - k$ more than 5, then $5k + 6(n - k) \le 6n - 12 \Longrightarrow k \ge 12$.
   c) No, e.g: icosahedron.
8. a) $2(3n - 6) \le n(n - 1)/2$ holds only if $n \ge 11$.
   b) E.g.: G is an 8-cycle with all the shortest diagonals and 2 longest diagonals.
9. At most 2: $e \le 3n - 6$. Adding 2 edges is possible.
10. Then $|E| = 3(n - 1) > 3n - 6 \Longrightarrow G$ cannot be planar.
11. Then $|E| = 2n > 2n - 4$, and all cycles are even $\Longrightarrow G$ cannot be planar.
12. a) $|E(K_8) = 28 = (3 \cdot 8 - 6) + 10$, and each „additional" edge creates a new crossing with the „planar" ones.
    b) $|E(K_{4,4}) = 16 = (2 \cdot 8 - 4) + 4 \Longrightarrow \exists \ge 4$ edge-crossings.
13. a) 205,
    b) 492.
14. b), f) and k) are planar, the rest are nonplanar.
15. G cannot contain a subgraph homomorphic to K_5 or $K_{3,3}$.
16. Yes, G cannot contain a subgraph homomorphic to K_5 or $K_{3,3}$.
17. a) A nonplanar graph has at least 9 edges.
    b) The complement of a K_5 subgraph contains $K_{3,3}$.