Exercise-set 5.
Solutions

1. Construct a graph G: $V(G)$ = people, and the edges are the acquaintances. Then $\deg(v) \geq 6 = 12/2$ $\Longrightarrow$ by Dirac's theorem $\exists$ a Hamilton cycle.
2. The condition in Ore's theorem holds for $G \Longrightarrow \exists$ a Hamilton cycle.
3. Construct a graph G: $V(G)$ = people, and the edges are the acquaintances. G is k-regular for some k. If $k \geq 10 \Longrightarrow G$ contains a Hamilton cycle, if $k \leq 9 \Longrightarrow \overline{G}$ contains a Hamilton cycle.
4. Construct a graph G: $V(G)$ = people, and the edges are the acquaintances. Then G contains no cycles of length 3 or 4. We need to show that $\exists$ a Hamilton cycle in G_1, where G_1 is obtained from G by adding edges between the second neighbors, i.e. and u and v are adjacent in $G_1 \Longleftrightarrow$ the 2 people know each other or they have a common friend. Then $\deg_{G_1}(v) \geq 5 + 5 \cdot 4 = 25$ (using the property of G) $\Longrightarrow$ by Dirac's theorem $\exists$ a Hamilton cycle in G_1.
5. A cycle on n vertices is like that (check).
6. E.g. K_7 with the edge $\{u, v\}$ missing and the 8th vertex is connected to u.
7. Add a new vertex to G, and connect it to all the old vertices. Then the new graph contains a Hamilton cycle from which we can get a Hamilton path of G.
8. Add two new non-adjacent vertices to G, and connect them to all the old vertices. Then the new graph contains a Hamilton cycle from which we can get a Hamilton path of G.
9. Delete v from G. Then the new graph contains a Hamilton cycle from which we can get a Hamilton path of G.
10. The 8 edges must have pairwise no common endpoints (i.e. be independent). Every second edge of a Hamilton cycle will do (which exists because $\deg(v) \geq n/2\ \forall v$).
11. We can add the edges of a Hamilton cycle of $\overline{G}$.
12. Need to add k pairwise non-adjacent edges (from $\overline{G}$). $\overline{G}$ contains a Hamilton cycle ($\deg_{\overline{G}}(v) = k,\ \forall v \in V(G)$). Every second edge of it will do.
13. Construct a graph G: $V(G)$ = four-element subsets, and they are adjacent, if they have at least two elements in common. Then G contains a Hamilton cycle.
14. The first graph is not bipartite (contains 5-cycles), but the second graph is.
15. Deleting 2 edges are enough, but less is not, since $\exists$ 2 edge-disjoint odd cycles in G.
16. The graph determined by the knights and attacks is bipartite (the two classes are to the white and black squares), and each of its degrees is at least 2 $\Longrightarrow$ $\exists$ a degree ≥ 3.
17. Yes (the two classes of vertices are sequences with an even or odd number of 1's, resp.).
18. No (the complement contains a triangle).
19. The vertices cannot be divided into two classes (degrees).
20. Complete bipartite graphs are like that.
21. The graphs are exactly the odd cycles (so in particular n must be odd). G must contain an odd cycle (otherwise $\chi(G') = 2$), and cannot contain more vertices or edges.
22. $\omega(G) = 3 \Longrightarrow \chi(G) \geq 3$, and G can be colored with 3 colors $\Longrightarrow \chi(G) \leq 3$.
23. $\omega(G) = 8$ (each row and column is a clique) $\Longrightarrow \chi(G) \geq 8$, and G can be colored with 8 colors (colors are diagonal) $\Longrightarrow \chi(G) \leq 8$.
24. G is bipartite (the two classes of vertices are the even and odd numbers, resp.) $\Longrightarrow \chi(G) = 2$
25. a), b) $\omega(G) = 3 \Longrightarrow \chi(G) \geq 3$, but G cannot be colored with 3 colors (proof!) $\Longrightarrow \chi(G) \geq 4$. G can be colored with 4 colors $\Longrightarrow \chi(G) \leq 4$.
26. $\omega(G) = 3 \Longrightarrow \chi(G) \geq 3$, but G cannot be colored with 3 colors (proof!) $\Longrightarrow \chi(G) \geq 4$. G can be colored with 4 colors $\Longrightarrow \chi(G) \leq 4$.

27. $\chi(G) \geq \lceil n/2 \rceil$ (at most 2 vertices can get the same color), and G can be colored with this many colors $\implies \chi(G) = \lceil n/2 \rceil$.

28. $\omega(G) = 10$ (any 10 consecutive numbers form a clique) $\implies \chi(G) \geq 10$, and G can be colored with 10 colors (periodically) $\implies \chi(G) \leq 10$.

29. $\omega(G) = 5$ ($\{1, 8, 15, 22, 29\}$ is a clique) $\implies \chi(G) \geq 5$, and G can be colored with 5 colors $\implies \chi(G) \leq 5$.

30. $\omega(G) = 11$ ($\{10, 11, \ldots, 20\}$ is a clique) $\implies \chi(G) \geq 11$, and G can be colored with 11 colors $\implies \chi(G) \leq 11$.

31. $\omega(G) = 4$ (the powers of 2 form a clique) $\implies \chi(G) \geq 4$, and G can be colored with 4 colors (using the same color between consecutive powers of 2) $\implies \chi(G) \leq 4$.

32. $\omega(G) = 11$ (prime numbers and 1 form a clique) $\implies \chi(G) \geq 11$, and G can be colored with 11 colors $\implies \chi(G) \leq 11$.

33. G is K_{10} with a perfect matching deleted. $\omega(G) = 5 \implies \chi(G) \geq 5$, and G can be colored with 5 colors $\implies \chi(G) \leq 5$.

34. No: $\chi(G) = 6$, but $\omega(G) = 5$.

35. a) There must be at least one edge between any 2 color classes.
    b) Otherwise we could put all the vertices of a color class into other color classes.

36. A proper coloring of G can be given by pairs of colors from G_1 and G_2.

37. Use the greedy coloring in the original (increasing) order of the vertices.

38. Order the vertices: first the exceptional ones, then the rest, and use the greedy coloring.

39. Use the greedy coloring in the decreasing order of the degrees.

40. Keep choosing vertices of degree at most 5 (in the remaining graph), then use the greedy coloring in the opposite order.