Exercise-set 4.
Solutions

1. Not possible; possible.
2. $|V(G)| = \binom{8}{2} = 28$, $\deg(v) = \binom{6}{2} = 15$ $\forall v \in V(G) \implies$ no Euler-circuit.
3. $|V(G)| = 2^4 = 16$, $\deg(v) = \binom{4}{2} = 6$ $\forall v \in V(G)$, but G is not connected $\implies$ no Euler-circuit.
4. $|V(G)| = 2^n$, $\deg(v) = n$ $\forall v \in V(G)$, and G is connected $\implies$ $\exists$ Euler-circuit if and only if n is even.
5. Construct a graph G: $V(G) =$ children, and u and v are adjacent $\iff$ not next to each other in the circle. This graph contains an Euler-circuit ($\deg(v) = 8$ $\forall v \in V(G)$, connected). Most number of passes = length of an Euler-circuit = $|E(G)| = 40$.
6. Construct a graph G: $V(G) =$ digits $= \{0, 1, \ldots, 9\}$, and u and v are adjacent $\iff$ $u + v \neq 9$. This graph contains an Euler-circuit ($\deg(v) = 8$ $\forall v \in V(G)$, connected) $\iff$ $\exists n$.
7. There are 8 vertices of odd degree $\implies$ $8/2 - 1 = 3$ climb-ups are needed .
8. a) Add k new edges ($\implies \exists$ Euler-circuit), then delete them.
   b) No: each trail eliminates ≤ 2 odd degrees from G.
9. 4 vertices have odd degrees, and not all of them are adjacent $\implies$ 1 edge is enough.
10. Equivalently: at least how many edges have to be deleted, s.t. the remaining graph contains an Euler-trail? At least 2 ($\exists$ 6 vertices of odd degree in G) $\implies$ length of a trail $= |E(G)| - 2 = 2 \cdot 4 + 5 \cdot 5 - 2 = 31$.
11. Yes: connected, and each degree is even.
12. There can be at most 2 components $\implies$ adding one edge can make it connected, and the degrees will be OK.
13. It's true for an Euler circuit, and each component of G contains one.
14. A circuit passes from S to the complement of S and back an even number of times.
15. Yes.
16. a) False (if we delete a cycle component).
    b) True: connected and the degrees are even.
    c) False, G' is not necessarily connected.
17. a) There is no Hamilton cycle: if we delete 4 vertices we get 5 components.
    There is a Hamilton path: draw.
    b) There is no Hamilton cycle: if we delete 2 vertices we get 3 components.
    There is a Hamilton path: draw.
    c) There is no Hamilton cycle: if we delete 4 vertices we get 5 components.
    There is a Hamilton path: draw.
18. a) Yes (draw); yes.
    b) No (delete 11 vertices); yes (draw).
19. a) No (delete the 9 vertices divisible by 3 or 5).
    b) No as well.
20. Yes, we can construct it recursively from $n = 2$.
21. a) If we delete 2 vertices we get 3 components $\implies$ need to add at least 1 edge. That is enough (draw).
    b) If we delete 2 vertices we get 4 components $\implies$ need to add at least 2 edges. That is enough (draw).
22. If we delete 1 vertex (the center) we get 100 components $\implies$ need to add at least 99 edges. That is enough (path).

23. a) Construct a graph G: $V(G) =$ squares, and the edges are the possible moves of the horse. This graph contains no Hamilton path: if we delete the 4 middle vertices we get 6 components.
b) Construct a graph G: $V(G) =$ squares, and the edges are the possible moves of the horse. This graph contains no Hamilton cycle: if we delete the 12 vertices we get 13 components.

24. By contradiction: if C doesn't contain all the vertices of G, then we could get a longer cycle.