Combinatorics and Graph Theory 1.

Exercise-set 3.
Solutions

1. a) $n_1 \cdot 1 + n_2 \cdot 2 + 5 \cdot 3 = 2(n_1 + n_2 + 5 - 1) \implies n_1 = 7.$
2. $2 \cdot 1 + (n-3) \cdot 2 + 1 \cdot d = 2(n-1) \implies d = 2.$
3. One of the degrees is 1. $d \cdot 9 + 92 \cdot 1 = 200 \implies d = 12.$
4. The tree has an even number of vertices.
5. $10(n-1) = \binom{n}{2} - (n-1) \implies n = 1$ or $n = 22$.
6. Necessary: $n - 1 = \binom{n}{2} - (n-1) \implies n = 1$ or $n = 4$. Both are possible.
7.
8. a) No,
   b) yes.
9. A graph is a spanning tree and 3 more edges, each of which forms a cycle with the tree.
10. The graph contains a cycle, of lenght at least 3.
11. The number of edges in a spanning forest is 17.
12. A degree one vertex in a spanning tree is like that.
13. a) Yes (2 triangles),
    b) No ($n - e + r = 2$).
14. No ($n - e + 2n = 2 \implies e = 3n - 2$, contradiction).
15. $n = 8,\ r = 10$.
16. $n = 20,\ r = 12,\ k \cdot r = 2e,\ n - e + r = 2 \implies k = 5$ (dodecahedron).
17. If there are a quadrangular and b octogonal faces, then $3n = 4a + 8b,\ 2e = 3n,\ r = a + b \implies a - b = 6$.
18. Like the proof of Corollary 1 (of Euler's theorem), but with equalities.
19. a) Otherwise $3n \leq e \leq 3n - 6$, contradiction.
    b) I k vertices have degree 5 and $n - k$ more than 5, then $5k + 6(n-k) \leq 6n - 12 \implies k \geq 12$.
    c) No, e.g: icosahedron.
20. a) $2(3n-6) \leq n(n-1)/2$ holds only if $n \geq 11$.
    b) E.g.: G is an 8-cycle with all the shortest diagonals and 2 longest diagonals.
21. At most 2: $e \leq 3n - 6$. Adding 2 edges is possible.
22. Then $|E| = 3(n-1) > 3n - 6 \implies G$ cannot be planar.
23. a) $|E(K_8) = 28 = (3 \cdot 8 - 6) + 10$, and each „additional" edge creates a new crossing with the „planar" ones.
    b) $|E(K_{4,4}) = 16 = (2 \cdot 8 - 4) + 4 \implies \exists \geq 4$ edge-crossings.
24. a) 205,
    b) 492.
25. b), f) and k) are planar, the rest are nonplanar.
26. G cannot contain a subgraph homomorphic to K_5 or $K_{3,3}$.
27. Yes, G cannot contain a subgraph homomorphic to K_5 or $K_{3,3}$.
28. a) A nonplanar graph has at least 9 edges.
    b) The complement of a K_5 subgraph contains $K_{3,3}$.