Exercise-set 8.

1. In a school the students elect several committees. A student can be a member on several committees. Now every committee wants to select a president from its members. Every member of a committee is eligible for presidency, but the committees don't want to share presidents (i.e., one person can be a president of at most one committee). When can this be attained?

2. a) In an Indian tribe there are 7 girls (A,B,...,G) and 6 boys (H,I,...,M) to be married. The chieftain made the table below about the possible couples. Can he find a wife for each of the boys?
b) G and L don't want to get married anymore. Solve the problem in this case as well.

	A	B	C	D	E	F	G
H		$\star$				$\star$	
I	$\star$	$\star$	$\star$	$\star$	$\star$		$\star$
J		$\star$			$\star$	$\star$	
K	$\star$		$\star$	$\star$		$\star$	$\star$
L					$\star$	$\star$	$\star$
M		$\star$			$\star$		

3. In a bipartite graph $G(A,B;E)$ the inequality $\deg(u) \geq \deg(v)$ holds for each pair of vertices $u \in A,\ v \in B$. Show that in this case G contains a matching covering A.

4. There are n couples on a hike. They want to distribute $2n$ different chocolate bars among themselves (so that everybody gets one). We know that everybody likes at least n kinds from the $2n$ types, and each kind of chocolate is liked by at least one person in each couple. Prove that the chocolate bars can be distributed in such a way that everybody gets a type that he/she likes.

5. Suppose that the bipartite graph G on $2n$ vertices has n vertices in both of its classes, and that the degree of each vertex of G is more that $\frac{n}{2}$. Show that G contains a perfect matching.

6. Each class of a bipartite graph contains exactly 5 vertices, and the degree of each vertex is at least 2. Show that this doesn't imply that the graph contains a perfect matching.

7. Let G be a simple, connected bipartite graph with n vertices in both of its vertex classes, and let all the degrees in one class be different. Show that G contains a perfect matching.

8. a) In a bipartite graph on 20 vertices 18 vertices have degree 5, and the degree of the other 2 vertices is 3. Show that the graph contains a perfect matching.
b) In a bipartite graph on 19 vertices 17 vertices have degree 6, and the degree of the other 2 vertices is 3. Show that the graph contains a matching of 9 edges.

9. Let the two vertex classes of the bipartite graph $G(A,B;E)$ be $A = \{a_1, a_2, \ldots, a_8\}$ and $B = \{b_1, b_2, \ldots, b_8\}$. For each $1 \leq i,j \leq 8$ let a_i and b_j be adjacent if the entry in the ith row and jth column of the matrix below is 1. Determine whether G contains a perfect matching or not.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

10. (MT+'19) Let the two vertex classes of the bipartite graph $G(A,B;E)$ be $A = \{a_1, a_2, \ldots, a_7\}$ and $B = \{b_1, b_2, \ldots, b_8\}$. For each $1 \leq i,j \leq 7$ let a_i and b_j be adjacent if the entry in the ith row and jth column of the matrix below is 1. Determine whether G contains a matching covering A or not.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

11. Prove that in a 2-regular bipartite graph the number of the different perfect matchings is always a power of 2.

12. Somebody selected 32 squares on a (8×8) chessboard in such a way that each row and each column contains exactly four selected squares. Show that we can select 8 out of the 32 squares in such a way that each row and each column contains exactly one of them.

13. $(*)$ Somebody divided a pack of 52 cards into 13 sets of 4 cards each at random. Prove that we can select one card from each set in such a way that we select exactly one of each figure.

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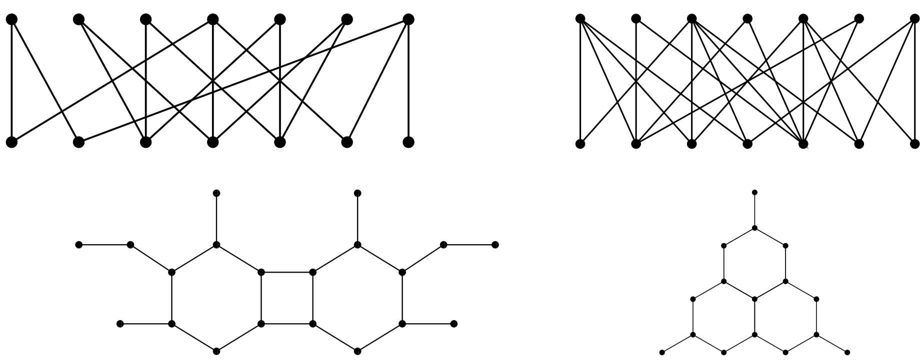
14. Let the two vertex classes of the bipartite graph $G(A, B; E)$ be $A = \{a_1, a_2, \ldots, a_9\}$ and $B = \{b_1, b_2, \ldots, b_9\}$. For each $1 \le i \le 9$ and $1 \le j \le 9$ let a_i and b_j be adjacent if the entry in the ith row and jth column of the matrix below is 1. Determine a maximum matching and a minimum covering set in G.

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

15. (MT'19) Let the two vertex classes of the bipartite graph $G(A, B; E)$ be $A = \{a_1, a_2, \ldots, a_{101}\}$ and $B = \{b_1, b_2, \ldots, b_{101}\}$. For each $1 \le i \le 101$ and $1 \le j \le 101$ let a_i and b_j be adjacent if $i \cdot j$ is even. Determine $\nu(G)$, the maximum number of independent edges, $\rho(G)$, the minimum number of covering edges, and give a maximum matching and a minimum covering set of edges in G.

16. Determine a maximum matching in each of the graphs below. Show that they are really maximum!



17. Let the vertex set of the simple graph be $V(G) = \{1, 2, \ldots, 10\}$. Let the vertices $x, y \in V(G)$ be adjacent if and only if $|x - y| = 3$ or $|x - y| = 5$. Delete the edge $\{3, 8\}$ from the graph G, and denote the graph obtained by H.
a) Determine $\nu(H)$, the maximum number of independent edges in H and determine a maximum matching in H.
b) Determine $\alpha(H)$, the maximum number of independent vertices in H and determine a maximum independent set of vertices in H.

18. Use Tutte's theorem to prove that the graph below doesn't contain a perfect matching. (Tutte's theorem gives a necessary and sufficient condition for an arbitrary graph to contain a perfect matching.)