1. In a school the students elect several committees. A student can be a member on several committees. Now every committee wants to select a president from its members. Every member of a committee is eligible for presidency, but the committees don’t want to share presidents (i.e., one person can be a president of at most one committee). When can this be attained?

2. a) In an Indian tribe there are 7 girls (A,B,...,G) and 6 boys (H,I,...,M) to be married. The chieftain made the table below about the possible couples. Can he find a wife for each of the boys?
   b) G and L don’t want to get married anymore. Solve the problem in this case as well.

   \[
   \begin{array}{cccccccc}
   H & I & J & K & L & M \\
   A & * & & & & \checkmark & \\
   B & * & * & * & * & * & \\
   C & & * & * & & & \\
   D & & & & * & * & \\
   E & & & & & & \\
   F & & & & & & \\
   G & & & & & & \\
   \end{array}
   \]

3. In a bipartite graph \(G(A,B;E)\) the inequality \(\deg(u) \geq \deg(v)\) holds for each pair of vertices \(u \in A, v \in B\). Show that in this case \(G\) contains a matching covering \(A\).

4. There are \(n\) couples on a hike. They want to distribute \(2n\) different chocolate bars among themselves (so that everybody gets one). We know that everybody likes at least \(n\) kinds from the \(2n\) types, and each kind of chocolate is liked by at least one person in each couple. Prove that the chocolate bars can be distributed in such a way that everybody gets a type that he/she likes.

5. Suppose that the bipartite graph \(G\) on \(2n\) vertices has \(n\) vertices in both of its classes, and that the degree of each vertex of \(G\) is more that \(n/2\). Show that \(G\) contains a perfect matching.

6. Each class of a bipartite graph contains exactly 5 vertices, and the degree of each vertex is at least 2. Show that this doesn’t imply that the graph contains a perfect matching.

7. Let \(G\) be a simple, connected bipartite graph with \(n\) vertices in both of its vertex classes, and let all the degrees in one class be different. Show that \(G\) contains a perfect matching.

8. a) In a bipartite graph on 20 vertices 18 vertices have degree 5, and the degree of the other 2 vertices is 3. Show that the graph contains a perfect matching.
   b) In a bipartite graph on 19 vertices 17 vertices have degree 6, and the degree of the other 2 vertices is 3. Show that the graph contains a matching of 9 edges.

9. Let the two vertex classes of the bipartite graph \(G(A,B;E)\) be \(A = \{a_1,a_2,\ldots,a_8\}\) and \(B = \{b_1,b_2,\ldots,b_8\}\). For each \(1 \leq i,j \leq 8\) let \(a_i\) and \(b_j\) be adjacent if the entry in the \(i\)th row and \(j\)th column of the matrix below is 1. Determine whether \(G\) contains a perfect matching or not.

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

10. (MT+’19) Let the two vertex classes of the bipartite graph \(G(A,B;E)\) be \(A = \{a_1,a_2,\ldots,a_7\}\) and \(B = \{b_1,b_2,\ldots,b_8\}\). For each \(1 \leq i,j \leq 7\) let \(a_i\) and \(b_j\) be adjacent if the entry in the \(i\)th row and \(j\)th column of the matrix below is 1. Determine whether \(G\) contains a matching covering \(A\) or not.

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\]
11. Prove that in a 2-regular bipartite graph the number of the different perfect matchings is always a power of 2.

12. Somebody selected 32 squares on a \((8 \times 8)\) chessboard in such a way that each row and each column contains exactly four selected squares. Show that we can select 8 out of the 32 squares in such a way that each row and each column contains exactly one of them.

13. (*) Somebody divided a pack of 52 cards into 13 sets of 4 cards each at random. Prove that we can select one card from each set in such a way that we select exactly one of each figure.

14. Let the two vertex classes of the bipartite graph \(G(A,B; E)\) be \(A = \{a_1, a_2, \ldots, a_9\}\) and \(B = \{b_1, b_2, \ldots, b_9\}\). For each \(1 \leq i \leq 9\) and \(1 \leq j \leq 9\) let \(a_i\) and \(b_j\) be adjacent if the entry in the \(i\)th row and \(j\)th column of the matrix below is 1. Determine a maximum matching and a minimum covering set in \(G\).

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

15. (MT’19) Let the two vertex classes of the bipartite graph \(G(A,B; E)\) be \(A = \{a_1, a_2, \ldots, a_{101}\}\) and \(B = \{b_1, b_2, \ldots, b_{101}\}\). For each \(1 \leq i \leq 101\) and \(1 \leq j \leq 101\) let \(a_i\) and \(b_j\) be adjacent if \(i \cdot j\) is even. Determine \(\nu(G)\), the maximum number of independent edges, \(\rho(G)\), the minimum number of covering edges, and give a maximum matching and a minimum covering set of edges in \(G\).

16. Determine a maximum matching in each of the graphs below. Show that they are really maximum!

17. Let the vertex set of the simple graph be \(V(G) = \{1, 2, \ldots, 10\}\). Let the vertices \(x, y \in V(G)\) be adjacent if and only if \(|x - y| = 3\) or \(|x - y| = 5\). Delete the edge \(\{3, 8\}\) from the graph \(G\), and denote the graph obtained by \(H\).

a) Determine \(\nu(H)\), the maximum number of independent edges in \(H\) and determine a maximum matching in \(H\).

b) Determine \(\alpha(H)\), the maximum number of independent vertices in \(H\) and determine a maximum independent set of vertices in \(H\).

18. Use Tutte’s theorem to prove that the graph below doesn’t contain a perfect matching. (Tutte’s theorem gives a necessary and sufficient condition for an arbitrary graph to contain a perfect matching.)