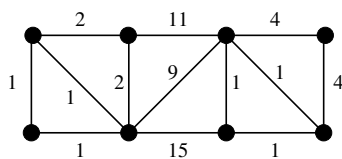


Exercise-set 4.

1. Determine a minimum weight spanning tree in the weighted graph below. How many such trees are there?



2. Let G be the complete graph on the vertex set $V(G) = \{1, 2, \dots, 100\}$. For every $1 \leq i, j \leq 100$, $i \neq j$ let the weight of the edge $\{i, j\}$ be the larger of the values of i and j . What is the weight of a minimum weight spanning tree in G ? Determine such a tree. How many minimum weight spanning trees are there?
3. Let G be the complete graph on the vertex set $V(G) = \{1, 2, \dots, 100\}$. For every $1 \leq i, j \leq 100$, $i \neq j$ let the weight of the edge $\{i, j\}$ be 1, if $i, j \leq 50$; 2, if $i, j \geq 51$; and 3 for all the other edges. What is the weight of a minimum weight spanning tree in G ? Determine such a tree.
4. Let G be a connected graph and $w : E(G) \rightarrow \mathbf{R}$ be a weight function on the edges of G . Suppose that one of the endpoints of the edge e of G is v and for all the edges f which are incident to v the inequality $w(e) \leq w(f)$ holds. Show that G has a minimum weight spanning tree which contains e .
5. Let G be a connected graph and $w : E(G) \rightarrow \mathbf{R}$ be a weight function on the edges of G . Furthermore, let C be a cycle in G and e an edge of C . Suppose that $w(e) \geq w(f)$ holds for all the edges f of the cycle C . Show that G has a minimum weight spanning tree which doesn't contain e .

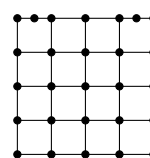
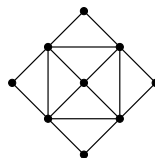
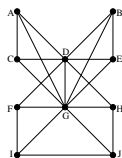
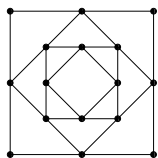
6. If it is possible, draw the figures below with one line, without lifting the pen.



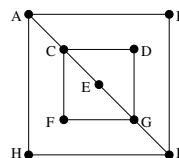
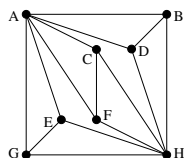
7. a) Let the vertices of the graph G be all the 2-element subsets of the 8-element set $S = \{a, b, c, d, e, f, g, h\}$, and two vertices be adjacent if and only if the corresponding subsets are disjoint. Does the graph G contain an Euler circuit?
- b) Let the vertices of the graph G be all the 3-element subsets of the 6-element set $S = \{a, b, c, d, e, f\}$, and two vertices be adjacent if and only if the corresponding subsets have at most one element in common. Does this graph G contain an Euler circuit?
8. Let the vertices of the graph G be all the 0-1 sequences of length 4, and two vertices be adjacent if and only if the corresponding sequences differ in exactly two digits. Does this graph G contain an Euler circuit?
9. 11 children play a game. They stand in a circle, and one of them starts passing a ball to somebody else, who in turn passes it on to a third child, etc. The rules are the following: nobody can throw the ball to somebody he/she has thrown it before, also nobody can throw the ball to somebody who has thrown it to him/her before, and nobody can throw the ball to either of the two children standing next to him/her in the circle. At most how many passes are possible in the game under these rules?
10. Show that there exists an integer n (in decimal form), in which the sum of the adjacent digits is never 9, but no matter how we choose two different integers between 0 and 9 whose sum is not 9, the two chosen numbers occur exactly once in n as adjacent digits (in some order).
11. In an imaginary language there are 10 vowels and 21 consonants. In this language there are no double letters (i.e. no letter can stand next to itself) and two different consonants cannot stand next to each other (either). But except for these anything else is possible, that is, any two different letters can stand next to each other if at least one of them is a vowel. What is the length of the longest sequence of letters in this language under the conditions that every letter can be used several times, but any two different letters can stand next to each other at most once in it?

12. In a connected graph G $2k$ ($k \geq 1$) vertices have odd degrees. Show that G is the union of k disjoint trails. Can it be the union of less than k trails?
13. The graph G on 10 vertices is constructed from two (vertex-disjoint) paths on 5 vertices in such a way that we connect each vertex of one path with every vertex of the other path. At most how many edges can a trail in G contain?
14. (MT'19) Determine the length of the longest trail in K_{10} , the complete graph on 10 vertices.
15. a) In a simple graph on 20 vertices the degree of each vertex is 6. Prove that we can add one new edge to the graph in such a way that the resulting graph is still simple and contains an Euler trail.
b) (MT+'20) In the simple graph G on 8 vertices there is no isolated vertex and the degree of each vertex is even. Show that we can add one edge to G in such a way that the graph obtained is still simple and contains an Euler trail.
16. For which of the values $r = 1, 2, \dots, 9$ does it hold that every simple r -regular graph on 10 vertices contains an Euler circuit? (A graph is r -regular if the degree of each of its vertices is r .)
17. In a simple graph with two components exactly four vertices have odd degrees. Moreover we know that neither of the components contain an Euler circuit. Is it always true that we can add two edges to the graph in such a way that the graph obtained is still simple and contains an Euler circuit?

18. Do the following graphs contain a Hamilton cycle? And a Hamilton path?



19. Let the vertices of the graph G be the squares of a 5×5 chessboard, and two vertices be adjacent if and only if the corresponding squares have a common edge. The graph G_1 is obtained from G by deleting a vertex corresponding to one of the corners of the chessboard from it (so G_1 has 24 vertices). The graph G_2 is obtained from G by deleting two vertices corresponding to opposite corners of the chessboard from it (so G_2 has 23 vertices).
a) Does G_1 contain a Hamilton cycle? And a Hamilton path?
b) Does G_2 contain a Hamilton cycle? And a Hamilton path?
20. Let the vertex set of the graph G be $V(G) = \{1, 2, \dots, 20\}$. Let the vertices $x, y \in V(G)$ be adjacent in G if $x \neq y$ and $x \cdot y$ is divisible by 3 or 5 (or both).
a) Does G contain a Hamilton path?
b) Does G contain a Hamilton cycle?
21. At least how many edges must be added to the graphs below so that the graphs obtained contain a Hamilton cycle?



22. The graph G is a star on 101 vertices (i.e. G has one vertex of degree 100 and hundred vertices of degree 1). At least how many edges must be added to G so that the graph obtained contains a Hamilton cycle?
23. a) Show that it is impossible to visit each square of a 4×4 chessboard (exactly once) with a horse.
b) Show that it is impossible to visit each square of a 5×5 chessboard (exactly once) with a horse such that in the 25th move we arrive back to the starting square.
c) (*) Can we visit each square of a 3×5 chessboard exactly once with a horse?
24. (MT'19) Can we place the numbers $1, 2, \dots, 15$ on a circle in such a way that the difference of the neighboring numbers in absolute value is at least 4 and at most 7?
25. (MT+'19) The degree sequence of the simple graph G is $3, 3, 4, 4, 4, 4, 8, 8, 8$. Does G contain a Hamilton cycle?