Combinatorics and Graph Theory 1. 26.02.2021.

Exercise-set 3.

1. a) In a graph on n vertices all the degrees are at least $\frac{n}{2}$. Does it follow that the graph is connected?
   b) And if we suppose that the graph is simple?
2. In a simple graph on 100 vertices each degree is at least 33. Show that we can add one edge to the graph in such a way that the resulting graph is connected.
3. In a simple graph on 23 vertices the degree of each vertex is at least 7. Show that no matter how we choose three vertices of the graph, there will be a path between two of them.
4. Show that for a simple graph G either G or its complement $\overline{G}$ is connected.
5. Let G be a simple graph and $v \in V(G)$ be a vertex of odd degree. Show that there is a path in G which starts at v and ends in a vertex of odd degree different from v.

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6. a) In a tree the degree of each vertex is 1 or 2 or 3. How many vertices of degree 1 are there if there are 5 vertices of degree 3?
   b) Draw two such trees in which the number of vertices of degree 2 are different.
7. Prove that in a tree on n vertices there cannot be exactly $n-3$ vertices of degree 2.
8. In a tree only two kinds of degrees occur, one of them 9 times, the other one 92 times. What are the two degrees?
9. Show that if all the degrees in a tree are odd, then the number of edges is also odd.
10. How many vertices does the tree T have if the number of its edges is exactly one-tenth of the number of edges of its complement?
11. Determine all the trees (on at least two vertices) which are isomorphic to their complement.
12. (MT'19) In a tree there are no vertices of degree 2 or 3. Prove that at least two-thirds of all the vertices have degree 1.
13. (MT+'19) How many pairwise non-isomorphic trees are there on 11 vertices which contain only two kinds of degrees?
14. ($*$) Prove the following: if T_1 and T_2 are two trees on the same vertex set and e_1 is an edge of T_1 then there is an edge e_2 of T_2 such that both $T_1 - e_1 + e_2$ and $T_2 - e_2 + e_1$ are trees.

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15. Let the vertex set of the simple graph be $V(G) = \{1, 2, \ldots, 10\}$. Let the vertices $x, y \in V(G)$, $x \neq y$ be adjacent if and only if $|x - y| \leq 2$. Does G have a spanning tree, which
    a) contains all the edges $\{x, y\}$ of G for which $x, y \leq 3$ holds;
    b) contains all the edges $\{x, y\}$ of G for which $|x - y| = 2$ holds?
16. A simple connected graph on 100 vertices has 102 edges. Show that the graph contains three pairwise different cycles. (Two cycles are different if their edge sets are not the same.)
17. A simple connected graph on 100 vertices has 100 edges. Show that the graph contains three pairwise different spanning trees. (Two spanning trees are different if their edge sets are not the same.)
18. A graph on 20 vertices has 18 edges and 3 components. Show that exactly two of its components are trees.
19. Show that every connected graph contains a vertex whose deletion doesn't disconnect the graph.

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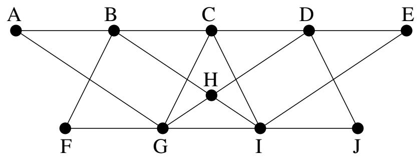
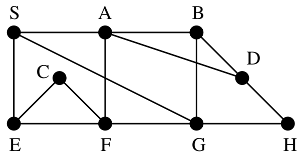
20. Can the vertices of the graph below be reached in the following order using the BFS algorithm?
    a) H, B, D, G, I, C, A, F, J, E   b) F, B, A, G, C, H, I, D, E, J
    c) J, D, I, C, E, G, H, A, F, B   d) A, B, G, C, H, F, I, D, E, J



21. The BFS algorithm visited the vertices of the graph below in the following order: $S, \square, \square, \square, H, \square, F, C, \square$.
a) Complete the sequence with the missing vertices (which are denoted by $\square$), and determine the corresponding BFS tree.
b) Can the edge $\{D, H\}$ be contained in an arbitrary BFS spanning tree started from S?

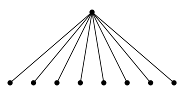


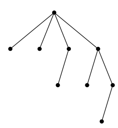
22. a) We want to decide for a given graph G and vertex s whether there is a cycle in G containing s and if yes then we want to find a shortest such cycle. How can we use the BFS algorithm to solve this problem?
b) And if we want to find a shortest cycle containing a given edge e?

23. In the connected graph G the degree of each vertex is 3. We start a BFS algorithm from vertex s which reaches vertex v in the 13th place (we consider s to be the vertex first reached). Is it possible that the distance of v from s is
a) 2, b) 3, c) 8?

24. We call the spanning tree F of a connected graph G *suitable* for a vertex v of G, if there is a BFS started from v which is exactly F. At most how many edges can a connected graph G on 100 vertices have if it has a spanning tree which is suitable for every vertex of G?

25. (MT+'19) Is it possible that we obtain the BFS spanning trees below started from two different vertices of the graph G?





26. (MT+'20) Let G be a simple undirected graph, and let a and b be two different vertices of it. Furthermore, we know that in a BFS started from vertex a the fifth vertex found is b. Is it true then that there is a BFS started from vertex b in which the fifth vertex found is a? (If the answer is yes, prove it, if no, give a counterexample.)