Exercise-set 9.

1. Determine the edge-chromatic number of the graph of the regular octahedron.
2. Show that $\chi_e(G) \geq e/\nu(G)$ holds for every loop-free graph G with e edges, where $\nu(G)$ is the size of a maximum set of independent edges (i.e. edges sharing no endpoints).
3. Determine $\chi_e(K_5)$ and $\chi_e(K_6)$. (In general, determine $\chi_e(K_{2n+1})$ and $\chi_e(K_{2n})$.)
4. Show that a championship with 20 participants can be arranged in 19 rounds. (Everybody plays with everybody else once, and in one round everybody can play at most once.)
5. Let G be the following graph: $V(G) = \{1, 2, \dots, 10\}$ és $E(G) = E_1 \cup E_2 \cup E_3$, ahol $E_1 = \{\{i, j\} : 1 \leq i < j \leq 5\}, E_2 = \{\{i, j\} : 6 \leq i < j \leq 10\}, E_3 = \{\{i, j\} : j = i+5\}$. (In words: G consists of two vertex-disjoint K_5 graphs, connected by a perfect matching.) Determine $\chi_e(G)$, the edge-chromatic number of the graph G.
6. Let G be a 10-regular simple graph on 1999 vertices. Determine $\chi_e(G)$, the edge-chromatic number of G.
7. The vertex v of the simple graph G has degree 2, but all the other vertices of G have degree 3. Determine $\chi_e(G)$, the edge-chromatic number of G.
   b) Let G be a 3-regular simple graph which contains a cut-edge (i.e. an edge whose deletion disconnects the graph). Show that $\chi_e(G) = 4$.
8. a) Let G be the graph obtained from a 5-cycle by substituting all of its edges by three parallel edges. Determine the edge-chromatic number of G.
   b) We double each edge of a cycle of length 5 of the complete graph on 5 vertices (i.e. we substitute the edges by two parallel edges). Determine the edge-chromatic number of the graph obtained.
9. (MT'19) The graph G on 15 vertices is constructed from two (vertex-disjoint) cycles on 7 and 8 vertices, respectively, in such a way that we connect each vertex of one cycle with every vertex of the other cycle. Determine $\chi_e(G)$, the edge-chromatic number of the graph G.
10. Let G be a k-regular graph with $\chi_e(G) = k$. Show that G contains a perfect matching.
11. In the simple graph on 9 vertices five vertices have degree 4, and the other four vertices have degree 3. Show that $\nu(G) = 4$.

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12. Somebody selected 30 squares on a 10×10 chessboard in such a way that each row and each column contains exactly three selected squares. We want to place 10 white, 10 black and 10 red stones on the 30 selected squares in such a way that each row and each column contains exactly one white, one black and one red stone. Prove that it is always possible with the given conditions.
13. We build a $4 \times 4 \times 4$ cube from 64 small cubes (so the length of an edge in the large cube is four times that of in the small cube). Let the vertices of the graph G be the small cubes, and two different vertices be adjacent if and only if the corresponding small cubes have a common face in the large cube. Determine $\chi_e(G)$, i.e. the edge-chromatic number of G.
14. The graph G on 10 vertices is constucted from two (vertex-disjoint) paths on 5 vertices in such a way that we connect each vertex of one path with every vertex of the other path. Determine $\chi_e(G)$, the edge-chromatic number of the graph G.
15. In a tree on 11 vertices each vertex has degree at most 3. Show that the tree has a matching of 4 edges.
16. (MT+'19) The graph G on 14 vertices is constructed from two (vertex-disjoint) cycles on 6 and 8 vertices, respectively, in such a way that we connect each vertex of one cycle with every vertex of the other cycle. Determine $\chi_e(G)$, the edge-chromatic number of the graph G.
17. (MT++'19) Let the vertices of the graph G be $v_1, v_2, \dots, v_{15}$, and let the vertices v_i and v_j be adjacent if and only if $|i-j| = 1, 5$ or 9. Determine $\chi_e(G)$, the edge-chromatic number of the graph G.
18. Show that if G is a simple k-regular graph on 9 vertices then $\chi_e(G) + \chi_e(\overline{G}) \geq 10$.

19. a) Show that if G is a 3-regular graph which contains a Hamilton cycle, then the edge-chromatic number of G is 3.
b) Show that the Petersen graph doesn't contain a Hamilton cycle.
c) $(*)$ Prove that if G is a 3-regular graph whose edges can be uniquely 3-colored (apart from the permutation of the colors), then G contains a Hamilton-cycle.

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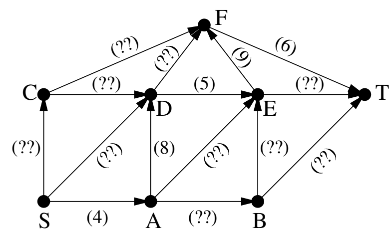
20. I just found a flow of value 15 in the network below when I spilled my coffee and the capacities of most of the edges became unreadable. Show that the flow I found was maximal.



21. Is it true that in the network (G, s, t, c) in the picture below the maximum flow value is exactly 19? (The numbers on the edges denote the appropriate capacities.)