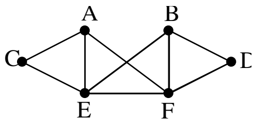
Combinatorics and Graph Theory 1.

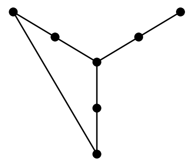
## Exercise-set 7.

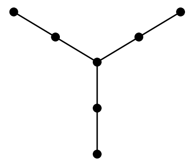
1. Show that in a loop-free graph $G$ on $n$ vertices the following hold, where $\alpha(G)$ denotes the size of a maximum independent set in $G$:
a) $\chi(G) \cdot \alpha(G) \geq n$,
b) $\chi(G) + \alpha(G) \leq n + 1$.

2. Is it true that every simple graph $G$ has a coloring with $\chi(G)$ colors in which one color class contains $\alpha(G)$ vertices?

3. Determine whether the following graphs are interval graphs or not. (A graph is an interval graph if the vertices of it are closed intervals on the real line, and and two vertices are adjacent if and only if the corresponding intervals intersect.)





4. Delete 4 edges from the complete graph on 8 vertices, in such a way that all of them are incident to a given vertex. Determine whether the graph obtained is an interval graph or not.

5. (MT++'19) Let $G$ be an interval graph. We construct the graph $G'$ from $G$ by adding a new vertex to it, and connecting it to all the vertices of $G$. Is $G'$ and interval graph or not?

6. Consider those intervals on the number line whose endpoints are both integers between 1 and 100, their length is at least 1 and at most 4, and at least one of their endpoints is an even number. Determine the chromatic number of the interval graph determined by them.

7. (MT+'19) Consider all the closed intervals on the real line with both endpoints in the set $\{1, 2, \ldots, 6\}$. Determine the chromatic number of the corresponding interval graph.

8. The chromatic number of an interval graph belonging to a given system of intervals is 10. Prove that if we delete a few intervals from the system in such a way that no three of them have a common point, then the interval graph belonging to the remaining intervals has chromatic number at least 8.

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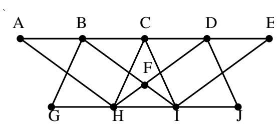
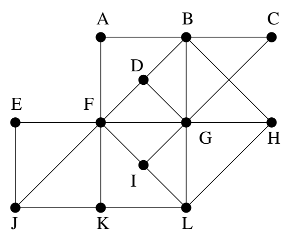
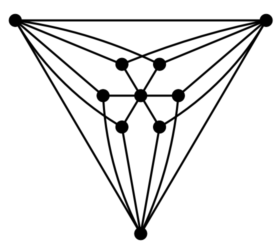
9. The vertices of an 18-vertex graph $G$ can be divided into 3 classes of six vertices each, in such a way that 2 vertices are adjacent if and only if they are in different classes. Determine the values $\chi(G), \nu(G), \tau(G), \alpha(G)$ and $\rho(G)$ for this graph $G$.

10. Determine the values $\nu(G), \tau(G), \alpha(G)$ and $\rho(G)$ for the graphs below, and determine a maximum matching, a maximum independent set (of vertices), and a minimum covering set of vertices and edges.



11. Let the vertices of $G$ be the numbers 1,2,...,2005, and two vertices, $i, j \in V$ be adjacent if and only if $i + j$ divided by 3 gives a remainder of 1. Determine the values $\chi(G), \nu(G), \tau(G), \alpha(G)$ and $\rho(G)$ for this graph $G$.

12. Let the vertex set of the graph $G$ be $V(G) = \{1, 2, \ldots, 60\}$. The vertices $x, y \in V(G)$ are adjacent in $G$ if $x \neq y$ and $x \cdot y$ is divisible by 6. Determine $\nu(G)$, i.e. the size of a maximum matching in $G$. (Hint: group the vertices according to the remainder when divided by 6.)

13. The vertices of a graph $G$ are the numbers 1,2,...,100, and two (different) vertices are adjacent if and only if their product is divisible by 7. Determine the the values $\alpha(G), \tau(G), \nu(G)$ and $\rho(G)$ for this graph $G$.

14. Let the vertices of the graph $G$ be the numbers $\{1, 2, \ldots, 100\}$, and two vertices, $i$ and $j$ be adjacent if and only if $i \neq j$ and the g.c.d. of $i$ and $j$ is even but not divisible by 4. Determine the parameters $\nu(G)$ and $\alpha(G)$. ($\nu(G)$ and $\alpha(G)$ denote the maximum number of independent edges and vertices, resp.)

15. (MT++'19) Let the vertices of the graph $G$ be $v_{i,j},\ 1 \leq i, j \leq 4$, and let the vertices $v_{i,j}$ and $v_{k,l}$ be adjacent if and only if $|i-k| + |j-l| = 1$. Determine the values $\nu(G), \tau(G), \rho(G), \alpha(G)$ for $G$.

16. a) Show that in a graph $G$ without loops the endpoints of a maximum matching $M$ form a covering set (of vertices).
b) Show that in a graph $G$ without loops $\tau(G) \leq 2\nu(G)$.
c) Show that in a graph $G$ without loops $\alpha(G) + 2 \cdot \nu(G) \geq |V(G)|$.

17. (MT'19) In the simple graph $G$ there is a subset $X$ of the vertices which is both an independent set and a covering set. Prove that in this case $G$ is a bipartite graph.

18. a) True or false: If a bipartite graph contains a Hamilton cycle then it contains a perfect matching?
b) Is the reverse of the stamement true or false?
c) Does the original statement remain true for non-bipartite graphs?

19. In the simple graph $G$ on $2k+1$ vertices the degree of each vertex is at least $k+1$. Determine $\nu(G)$, the maximum number of independent edges in $G$.

20. Let $G$ be a simple graph on $2n$ vertices. We know that two nonadjacent vertices, $u$ and $v$ have degree $n-1$, but all the other vertices of $G$ have degree at least $n$ (where $n > 1$ is an integer). Show that $G$ contains a perfect matching!

21. In a simple graph on 20 vertices the degree of each vertex is at least 9. Prove that the graph contains a matching of 9 edges.

22. Let $M$ be an $n \times n$ matrix. Construct a bipartite graph $G$ from $M$ in the following way: let the two vertex classes of $G$ be $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, and for each $1 \leq i, j \leq n$ let $a_i$ and $b_j$ be adjacent if and only if the entry in the $i$th row and $j$th column of $M$ is not zero. Show that if $\det M \neq 0$ then there is a perfect matching in $G$.

23. At most how many edges can a simple graph $G$ on 100 points have if $\tau(G) \leq 20$? (The parameter $\tau(G)$ denotes the minimum number of covering points of $G$.)

24. Let the vertices of the graph $G$ be the numbers $1, 2, \ldots, 100$, and two vertices, $i$ and $j$ be adjacent if one of them divides the other. Determine the value of $\tau(G)$.

25. Show that in a a simple graph on $n$ vertices $\tau(G) = n-1$ if and only if $G = K_n$.

26. Prove that $\Delta(G) \cdot \tau(G) \geq |E(G)|$ for any graph $G$, where $\Delta(G)$ denotes tha maximum degree in $G$.