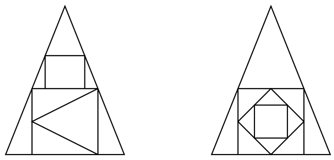
Combinatorics and Graph Theory 1.

Exercise-set 5.

1. If it is possible, draw the figures below with one line, without lifting the pen.



2. Let the vertices of the graph G be all the 2-element subsets of the 8-element set $S = \{a, b, c, d, e, f, g, h\}$, and two vertices be adjacent if and only if the corresponding subsets are disjoint. Does the graph G contain an Euler circuit?

3. Let the vertices of the graph G be all the 0-1 sequences of length 4, and two vertices be adjacent if and only if the corresponding sequences differ in exactly two digits. Does this graph G contain an Euler circuit?

4. Let the vertices of the graph G_n be all the 0-1 sequences of length n, and two vertices be adjacent if and only if the corresponding sequences differ in exactly one digit. For which values of n does the graph G_n contain an Euler circuit?

5. 11 children play a game. They stand in a circle, and one of them starts passing a ball to somebody else, who in turn passes it on to a third child, etc. The rules are the following: nobody can throw the ball to somebody he/she has thrown it before, also nobody can throw the ball to somebody who has thrown it to him/her before, and nobody can throw the ball to either of the two children standing next to him/her in the circle. At most how many passes are possible in the game under these rules?

6. Show that there exists an integer n (in decimal form), in which the sum of the adjacent digits is never 9, but no matter how we choose two different integers between 0 and 9 whose sum is not 9, the two chosen numbers occur exactly once in n as adjacent digits (in some order).

7. In a connected graph G $2k$ $(k \geq 1)$ vertices have odd degrees. Show that G is the union of k disjoint trails. Can it be the union of less than k trails?

8. The graph G on 10 vertices is constucted from two (vertex-disjoint) paths on 5 vertices in such a way that we connect each vertex of one path with every vertex of the other path. At most how many edges can a trail in G contain?

9. The graph G contains a circuit which contains every edge of G an odd number of times. Does it follow that G contains an Euler-circuit as well?

10. In a simple graph on 20 vertices the degree of each vertex is 6. Prove that we can add one new edge to the graph in such a way that the resulting graph is still simple and contains an Euler trail.

11. Prove that if the graph G contains an Euler circuit, then from every subset of vertices S there is an even number of edges going to the complement of S.

12. We know that the graph G contains an Euler circuit, and let e and f be two adjacent edges of G. Is it true that G contains an Euler circuit in which e and f follow each other?

13. Decide whether the statements below are true or not.
a) If we delete the edges of one cycle from the graph G and the remaining graph G' contains an Euler circuit, then G contains one as well.
b) If G is connected and we delete the edges of one cycle from G and the remaining graph G' contains an Euler circuit, then G contains one as well.
c) I G contains an Euler circuit and we delete the edges of one cycle from G then the remaining graph G' contains as well.

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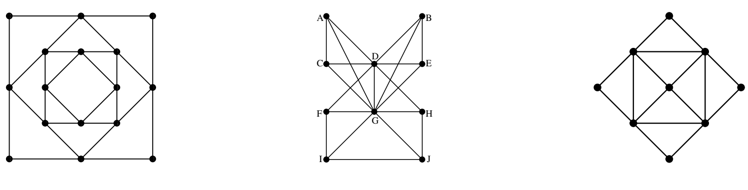
14. Do the following graphs contain a Hamilton cycle? And a Hamilton path?

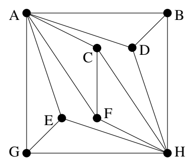


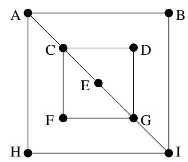
15. Let the vertices of the graph G be the squares of a 5×5 chessboard, and two vertices be adjacent if and only if the corresponding squares have a common edge. The graph G_1 is obtained from G by deleting a vertex corresponding to one of the corners of the chessboard from it (so G_1 has 24 vertices). The graph G_2 is obtained from G by deleting two vertices corresponding to opposite corners of the chessboard from it (so G_2 has 23 vertices).
a) Does G_1 contain a Hamilton cycle? And a Hamilton path?
b) Does G_2 contain a Hamilton cycle? And a Hamilton path?

16. Let the vertex set of the graph G be $V(G) = \{1, 2, \ldots, 20\}$. Let the vertices $x, y \in V(G)$ be adjacent in G if $x \neq y$ and $x \cdot y$ is divisible by 3 or 5 (or both).
a) Does G contain a Hamilton path?
b) Does G contain a Hamilton cycle?

17. Let the vertices of the graph G_n be all the 0-1 sequences of length n, and two vertices be adjacent if and only if the corresponding sequences differ in exactly one digit. Does the graph G_n contain a Hamilton cycle?

18. At least how many edges must be added to the graphs below so that the graphs obtained contain a Hamilton cycle?





19. The graph G is a star on 101 vertices (i.e. G has one vertex of degree 100 and hundred vertices of degree 1). At least how many edges must be added to G so that the graph obtained contains a Hamilton cycle?

20. a) Show that it is impossible to visit each square of a 4×4 chessboard (exactly once) with a horse.
b) Show that it is impossible to visit each square of a 5×5 chessboard (exactly once) with a horse such that in the 25th move we arrive back to the starting square.

21. Let G be a connected graph and let C be a cycle in G, for which it holds that if we delete any edge from it then we get a longest path in G. Prove that in this case C is a Hamilton cycle of G.

22. In a company of 12 everybody knows at least 6 others (acquaintances are mutual). Show that this company can be seated around a round table in such a way that everybody knows his/her neighbors.

23. In a company of 20 everybody knows the same number of people (acquaintances are mutual). Show that this company can be seated around a round table in such a way that either everybody knows his/her neighbors or nobody knows his or her neighbors.

24. There are 50 guests at a banquet, each of them knows at least 5 people from the others. (Acquaintances are mutual.) No matter how we choose 3 or 4 from the guests they cannot sit down to a round table in such a way that everybody knows both of his/her neighbors. Show that in this case all the guests can be seated around a round table for 50 persons in such a way that any two people who sit next to each other, but don't know each other have a common friend among the guests.

25. a) Show that for each $n \geq 5$ there is a graph G on n vertices such that both G and its complement contain a Hamilton cycle.
b) Give a simple, connected graph G on 8 vertices, whose complement is also connected, and neither G nor its complement contain a Hamilton cycle.

26. In the simple graph G on $2k + 1$ vertices each vertex has degree at least k. Prove that G contains a Hamilton path.

27. In a simple graph on 20 vertices the degree of each vertex is at least 9. Prove that we can add one new edge to the graph in such a way that the resulting graph contains a Hamilton path.

28. In the simple graph G on 201 vertices the degree of each vertex, except for v, is at least 101. About v we only know that it is not an isolated vertex. Show that G contains a Hamilton path.

29. Show that if G is a simple 9-regular graph on 16 vertices, then we can delete 8 edges of G in such a way that the remaining graph contains an Euler circuit.

30. In a simple graph on 20 vertices the degree of each vertex is 8. Prove that we can add 20 new edges to the graph in such a way that the resulting graph is still simple and contains an Euler circuit.

31. Let G be a simple graph on $2k$ vertices in which the degree of each vertex is $k - 1$, where $k > 1$ is an integer. Prove that we can add k new edges to G in such a way that the resulting graph contains a Hamilton cycle.