1. If it is possible, draw the figures below with one line, without lifting the pen.

2. Let the vertices of the graph $G$ be all the 2-element subsets of the 8-element set $S = \{a, b, c, d, e, f, g, h\}$, and two vertices be adjacent if and only if the corresponding subsets are disjoint. Does the graph $G$ contain an Euler circuit?

3. Let the vertices of the graph $G$ be all the 0-1 sequences of length 4, and two vertices be adjacent if and only if the corresponding sequences differ in exactly two digits. Does this graph $G$ contain an Euler circuit?

4. Let the vertices of the graph $G_n$ be all the 0-1 sequences of length $n$, and two vertices be adjacent if and only if the corresponding sequences differ in exactly one digit. For which values of $n$ does the graph $G_n$ contain an Euler circuit?

5. 11 children play a game. They stand in a circle, and one of them starts passing a ball to somebody else, who in turn passes it on to a third child, etc. The rules are the following: nobody can throw the ball to somebody he/she has thrown it before, also nobody can throw the ball to somebody who has thrown it to him/her before, and nobody can throw the ball to either of the two children standing next to him/her in the circle. At most how many passes are possible in the game under these rules?

6. Show that there exists an integer $n$ (in decimal form), in which the sum of the adjacent digits is never 9, but no matter how we choose two different integers between 0 and 9 whose sum is not 9, the two chosen numbers occur exactly once in $n$ as adjacent digits (in some order).

7. In a connected graph $G_{2k}$ ($k \geq 1$) vertices have odd degrees. Show that $G$ is the union of $k$ disjoint trails. Can it be the union of less than $k$ trails?

8. The graph $G$ on 10 vertices is constructed from two (vertex-disjoint) paths on 5 vertices in such a way that we connect each vertex of one path with every vertex of the other path. At most how many edges can a trail in $G$ contain?

9. The graph $G$ contains a circuit which contains every edge of $G$ an odd number of times. Does it follow that $G$ contains an Euler-circuit as well?

10. In a simple graph on 20 vertices the degree of each vertex is 6. Prove that we can add one new edge to the graph in such a way that the resulting graph is still simple and contains an Euler trail.

11. Prove that if the graph $G$ contains an Euler circuit, then from every subset of vertices $S$ there is an even number of edges going to the complement of $S$.

12. We know that the graph $G$ contains an Euler circuit, and let $e$ and $f$ be two adjacent edges of $G$. Is it true that $G$ contains an Euler circuit in which $e$ and $f$ follow each other?

13. Decide whether the statements below are true or not.
   a) If we delete the edges of one cycle from the graph $G$ and the remaining graph $G'$ contains an Euler circuit, then $G$ contains one as well.
   b) If $G$ is connected and we delete the edges of one cycle from $G$ and the remaining graph $G'$ contains an Euler circuit, then $G$ contains one as well.
   c) If $G$ contains an Euler circuit and we delete the edges of one cycle from $G$ then the remaining graph $G'$ contains as well.

14. Do the following graphs contain a Hamilton cycle? And a Hamilton path?
15. Let the vertices of the graph $G$ be the squares of a $5 \times 5$ chessboard, and two vertices be adjacent if and only if the corresponding squares have a common edge. The graph $G_1$ is obtained from $G$ by deleting a vertex corresponding to one of the corners of the chessboard from it (so $G_1$ has 24 vertices). The graph $G_2$ is obtained from $G$ by deleting two vertices corresponding to opposite corners of the chessboard from it (so $G_2$ has 23 vertices). 
   a) Does $G_1$ contain a Hamilton cycle? And a Hamilton path? 
   b) Does $G_2$ contain a Hamilton cycle? And a Hamilton path?

16. Let the vertex set of the graph $G$ be $V(G) = \{1, 2, \ldots, 20\}$. Let the vertices $x, y \in V(G)$ be adjacent in $G$ if $x \neq y$ and $x \cdot y$ is divisible by 3 or 5 (or both). 
   a) Does $G$ contain a Hamilton path? 
   b) Does $G$ contain a Hamilton cycle?

17. Let the vertices of the graph $G_n$ be all the 0-1 sequences of length $n$, and two vertices be adjacent if and only if the corresponding sequences differ in exactly one digit. Does the graph $G_n$ contain a Hamilton cycle?

18. At least how many edges must be added to the graphs below so that the graphs obtained contain a Hamilton cycle?

19. The graph $G$ is a star on 101 vertices (i.e. $G$ has one vertex of degree 100 and hundred vertices of degree 1). At least how many edges must be added to $G$ so that the graph obtained contains a Hamilton cycle?

20. a) Show that it is impossible to visit each square of a $4 \times 4$ chessboard (exactly once) with a horse. 
   b) Show that it is impossible to visit each square of a $5 \times 5$ chessboard (exactly once) with a horse such that in the 25th move we arrive back to the starting square.

21. Let $G$ be a connected graph and let $C$ be a cycle in $G$, for which it holds that if we delete any edge from it then we get a longest path in $G$. Prove that in this case $C$ is a Hamilton cycle of $G$.

22. In a company of 12 everybody knows at least 6 others (acquaintances are mutual). Show that this company can be seated around a round table in such a way that everybody knows his/her neighbors.

23. In a company of 20 everybody knows the same number of people (acquaintances are mutual). Show that this company can be seated around a round table in such a way that everybody knows his/her neighbors or nobody knows his or her neighbors.

24. There are 50 guests at a banquet, each of them knows at least 5 people from the others. (Acquaintances are mutual.) No matter how we choose 3 or 4 from the guests they cannot sit down to a round table in such a way that everybody knows both of his/her neighbors. Show that in this case all the guests can be seated around a round table for 50 persons in such a way that any two people who sit next to each other, but don’t know each other have a common friend among the guests.

25. a) Show that for each $n \geq 5$ there is a graph $G$ on $n$ vertices such that both $G$ and its complement contain a Hamilton cycle. 
   b) Give a simple, connected graph $G$ on 8 vertices, whose complement is also connected, and neither $G$ nor its complement contain a Hamilton cycle.

26. In the simple graph $G$ on $2k + 1$ vertices each vertex has degree at least $k$. Prove that $G$ contains a Hamilton path.

27. In a simple graph on 20 vertices the degree of each vertex is at least 9. Prove that we can add one new edge to the graph in such a way that the resulting graph contains a Hamilton path.

28. In the simple graph $G$ on 201 vertices the degree of each vertex, except for $v$, is at least 101. About $v$ we only know that it is not an isolated vertex. Show that $G$ contains a Hamilton path.

29. Show that if $G$ is a simple 9-regular graph on 16 vertices, then we can delete 8 edges of $G$ in such a way that the remaining graph contains an Euler circuit.

30. In a simple graph on 20 vertices the degree of each vertex is 8. Prove that we can add 20 new edges to the graph in such a way that the resulting graph is still simple and contains an Euler circuit.

31. Let $G$ be a simple graph on $2k$ vertices in which the degree of each vertex is $k - 1$, where $k > 1$ is an integer. Prove that we can add $k$ new edges to $G$ in such a way that the resulting graph contains a Hamilton cycle.