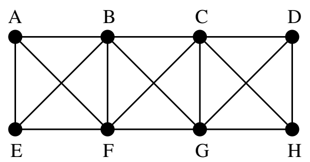
Combinatorics and Graph Theory 1.

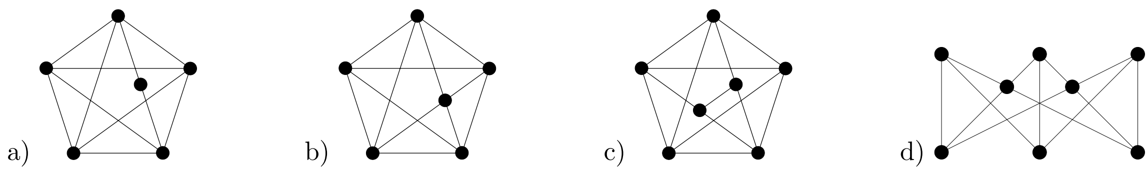
Exercise-set 4.

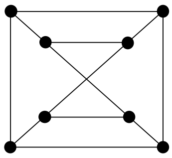
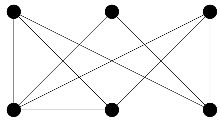
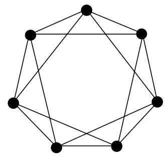
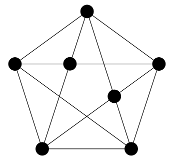
1. a) Is there a plane graph in which the number of vertices, edges and regions are all divisible by 3?
   b) Is there a connected graph with the above properties?
2. Is there a simple connected plane graph which has half as many vertices as regions?
3. In a simple, connected plane graph the degree of each vertex is 4, and the number of the edges is 16. Determine the number of regions of the drawing.
4. A convex polyhedron has 20 vertices and 12 faces. Each face of the polyhedron is bounded by the same number of edges. What is this common number?
5. Each face of a convex polyhedron is a (not necessarily regular) quadrangle or an octogon. Furthermore we know that exactly 3 faces meet at each vertex of the polyhedron. What is the difference of the number of quadrangular and octogonal faces?
6. Let G be a simple, connected plane graph on $n \geq 3$ vertices, all of whose regions are triangles. Show that G has exactly $3n - 6$ edges. Draw such a graph.
7. a) Show that in a simple planar graph the minimum degree is at most 5.
   b) In the simple planar graph G the minimum degree is 5. Show that in this case G contains at least 12 vertices of degree 5.
   c) Is the statement true with 13 instead of 12?
8. a) Pove that if G is a simple planar graph on at least 11 vertices then at least one of G and its complement $\overline{G}$ is not planar.
   b) Give a simple graph G on 8 vertices such that both G and $\overline{G}$ are planar.
9. At most how many edges can be added to the graph below in such a way that we get a simple planar graph? (We add edges only between already existing vertices.)

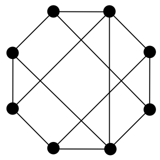


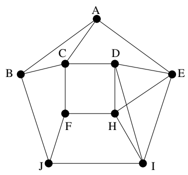
10. Suppose that $G = (V, E)$ is a simple graph whose set of edges E is the union of the disjoint sets of edges E_1, E_2 and E_3, where all the three subgraphs (V, E_1), (V, E_2) and (V, E_3) are spanning trees of G. Show that in this case G is not planar.
11. (MT+'19) Let G be a simple 4-regular bipartite graph (i.e. in which the degree of each vertex is 4). Can G be a planar graph?
12. a) Show that in a drawing of K_8 in which three edges cannot go through a common point there are at least 10 edge-crossings.
    b) At least how many edge-crossings are there in a drawing of $K_{4,4}$ if three edges cannot go through a common point?
13. a) Determine the maximum of $n(G) + e(G) - 2r(G)$ over all connected simple plane graphs on 100 vertices.
    b) Determine the maximum of $n(G) + 2e(G) - r(G)$ over all connected simple plane graphs on 100 vertices.

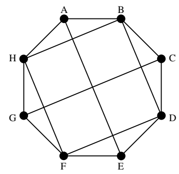
14. Decide whether the following graphs are planar or not:

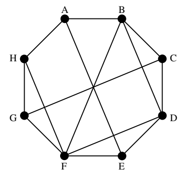


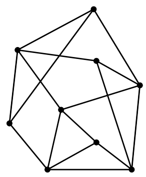
e)  f)  g)  h) 

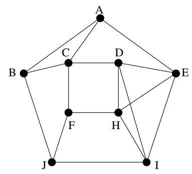
i) 

j) 

k) 

l) 

m) (MT'19) 

n) (MT++'19) 

15. In the graph G the degree of each vertex is at most 3. Furthermore we know that each cycle of G contains at most 5 edges. Show that G is a planar graph.

16. The graph G doesn't contain a subgraph homeomorphic to $K_{2,3}$ (the complete bipartite graph on 2+3 vertices). Does it follow that G is planar?

17. a) Let G be a simple graph on at most 6 vertices. Pove that at least one of G and its complement $\overline{G}$ is planar.
b) Give a simple graph G on 8 vertices such that neither G nor $\overline{G}$ is planar.