Exercise-set 4.

1. a) Is there a plane graph in which the number of vertices, edges and regions are all divisible by 3?  
b) Is there a connected graph with the above properties?

2. Is there a simple connected plane graph which has half as many vertices as regions?

3. In a simple, connected plane graph the degree of each vertex is 4, and the number of the edges is 16. Determine the number of regions of the drawing.

4. A convex polyhedron has 20 vertices and 12 faces. Each face of the polyhedron is bounded by the same number of edges. What is this common number?

5. Each face of a convex polyhedron is a (not necessarily regular) quadrangle or an octagon. Furthermore we know that exactly 3 faces meet at each vertex of the polyhedron. What is the difference of the number of quadrangular and octogonal faces?

6. Let \( G \) be a simple, connected plane graph on \( n \geq 3 \) vertices, all of whose regions are triangles. Show that \( G \) has exactly \( 3n - 6 \) edges. Draw such a graph.

7. a) Show that in a simple planar graph the minimum degree is at most 5.  
b) In the simple planar graph \( G \) the minimum degree is 5. Show that in this case \( G \) contains at least 12 vertices of degree 5.  
c) Is the statement true with 13 instead of 12?

8. a) Prove that if \( G \) is a simple planar graph on at least 11 vertices then at least one of \( G \) and its complement \( \overline{G} \) is not planar.  
b) Give a simple graph \( G \) on 8 vertices such that both \( G \) and \( \overline{G} \) are planar.

9. At most how many edges can be added to the graph below in such a way that we get a simple planar graph? (We add edges only between already existing vertices.)

10. Suppose that \( G = (V,E) \) is a simple graph whose set of edges \( E \) is the union of the disjoint sets of edges \( E_1, E_2 \) and \( E_3 \), where all the three subgraphs \((V,E_1), (V,E_2)\) and \((V,E_3)\) are spanning trees of \( G \). Show that in this case \( G \) is not planar.

11. (MT+’19) Let \( G \) be a simple 4-regular bipartite graph (i.e. in which the degree of each vertex is 4). Can \( G \) be a planar graph?

12. a) Show that in a drawing of \( K_8 \) in which three edges cannot go through a common point there are at least 10 edge-crossings.  
b) At least how many edge-crossings are there in a drawing of \( K_{4,4} \) if three edges cannot go through a common point?

13. a) Determine the maximum of \( n(G) + e(G) - 2r(G) \) over all connected simple plane graphs on 100 vertices.  
b) Determine the maximum of \( n(G) + 2e(G) - r(G) \) over all connected simple plane graphs on 100 vertices.

14. Decide whether the following graphs are planar or not:
15. In the graph $G$ the degree of each vertex is at most 3. Furthermore we know that each cycle of $G$ contains at most 5 edges. Show that $G$ is a planar graph.

16. The graph $G$ doesn’t contain a subgraph homeomorphic to $K_{2,3}$ (the complete bipartite graph on 2+3 vertices). Does it follow that $G$ is planar?

17. a) Let $G$ be a simple graph on at most 6 vertices. Prove that at least one of $G$ and its complement $\overline{G}$ is planar.
   b) Give a simple graph $G$ on 8 vertices such that neither $G$ nor $\overline{G}$ is planar.