Exercise-set 3.

1. a) In a tree the degree of each vertex is 1 or 2 or 3. How many vertices of degree 1 are there if there are 5 vertices of degree 3?
   b) Draw two such trees in which the number of vertices of degree 2 are different.
2. Prove that in a tree on n vertices there cannot be exactly $n-3$ vertices of degree 2.
3. In a tree only two kinds of degrees occur, one of them 9 times, the other one 92 times. What are the two degrees?
4. Show that if all the degrees in a tree are odd, then the number of edges is also odd.
5. How many vertices does the tree T have if the number of its edges is exactly one-tenth of the number of edges of its complement?
6. Determine all the trees (on at least two vertices) which are isomorphic to their complement.
7. $(*)$ Prove the following: if T_1 and T_2 are two trees on the same vertex set and e_1 is an edge of T_1 then there is an edge e_2 of T_2 such that both $T_1 - e_1 + e_2$ and $T_2 - e_2 + e_1$ are trees.
8. (MT'19) In a tree there are no vertices of degree 2 or 3. Prove that at least two-thirds of all the vertices have degree 1.
9. (MT+'19) How many pairwise non-isomorphic trees are there on 11 vertices which contain only two kinds of degrees?

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10. Let the vertex set of the simple graph be $V(G) = \{1, 2, \ldots, 10\}$. Let the vertices $x, y \in V(G)$, $x \neq y$ be adjacent if and only if $|x - y| \leq 2$. Does G have a spanning tree, which
    a) contains all the edges $\{x, y\}$ of G for which $x, y \leq 3$ holds;
    b) contains all the edges $\{x, y\}$ of G for which $|x - y| = 2$ holds?
11. A simple connected graph on 100 vertices has 102 edges. Show that the graph contains three pairwise different cycles. (Two cycles are different if their edge sets are not the same.)
12. A simple connected graph on 100 vertices has 100 edges. Show that the graph contains three pairwise different spanning trees. (Two spanning trees are different if their edge sets are not the same.)
13. A graph on 20 vertices has 18 edges and 3 components. Show that exactly two of its components are trees.
14. Show that every connected graph contains a vertex whose deletion doesn't disconnect the graph.

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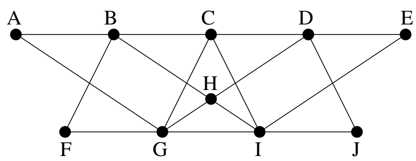
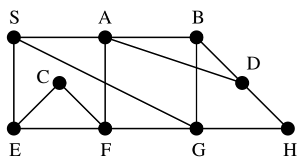
15. Can the vertices of the graph below be reached in the following order using the BFS algorithm?
    a) H, B, D, G, I, C, A, F, J, E   b) F, B, A, G, C, H, I, D, E, J
    c) J, D, I, C, E, G, H, A, F, B   d) A, B, G, C, H, F, I, D, E, J



16. The BFS algorithm visited the vertices of the graph below in the following order:
    $S, \square, \square, \square, H, \square, F, C, \square$.
    a) Complete the sequence with the missing vertices (which are denoted by $\square$), and determine the corresponding BFS tree.
    b) Can the edge $\{D, H\}$ be contained in an arbitrary BFS spanning tree started from S?



17. We want to decide for a given graph G and vertex s whether there is a cycle in G containing s and if yes then we want to find a shortest such cycle. Modify the BFS algorithm so that it can solve this problem as well.

18. In the connected graph G the degree of each vertex is 3. We start a BFS algorithm from vertex s which reaches vertex v in the 13th place (we consider s to be the vertex first reached). Is it possible that the distance of v from s is
a) 2, b) 3, c) 8?

19. We call the spanning tree F of a connected graph G *suitable* for a vertex v of G, if there is a BFS started from v which is exactly F. At most how many edges can a connected graph G on 100 vertices have if it has a spanning tree which is suitable for every vertex of G?

20. (MT+'19) Is it possible that we obtain the BFS spanning trees below started from two different vertices of the graph G?

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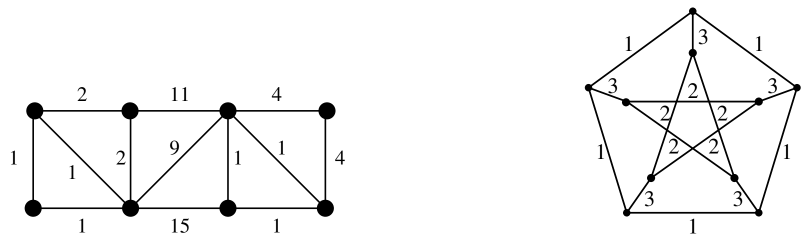
21. (-, MT'19) Determine a minimum weight spanning tree in each of the weighted graphs below. How many such trees are there?



22. Let G be the complete graph on the vertex set $V(G) = \{1, 2, \ldots, 100\}$. For every $1 \le i, j \le 100$, $i \ne j$ let the weight of the edge $\{i, j, \}$ be the larger of the values of i and j. What is the weight of a minimum weight spanning tree in G? Determine such a tree. How many minimum weight spanning trees are there?

23. Let G be the complete graph on the vertex set $V(G) = \{1, 2, \ldots, 100\}$. For every $1 \le i, j \le 100$, $i \ne j$ let the weight of the edge $\{i, j, \}$ be 1, if $i, j \le 50$, 2, if $i, j \ge 51$, and 3 for all the other edges. What is the weight of a minimum weight spanning tree in G? Determine such a tree.

24. Let G be a connected graph and $w : E(G) \to \mathbf{R}$ be a weight function on the edges of G. Suppose that one of the endpoints of the edge e of G is v and for all the edges f which are incident to v the inequality $w(e) \le w(f)$ holds. Show that G has a minimum weight spanning tree which contains e.

25. Let G be a connected graph and $w : E(G) \to \mathbf{R}$ be a weight function on the edges of G. Furthermore, let C be a cycle in G and e an edge of C. Suppose that $w(e) \ge w(f)$ holds for all the edges f of the cycle C. Show that G has a minimum weight spanning tree which doesn't contain e.

26. Let G be a connected weighted graph. Show that all the minimum weight spanning trees of G can be obtained by Kruskal's algorithm.

27. (MT++'19) Determine a minimum weight spanning tree in the graph below for all the values of the positive real parameter p.