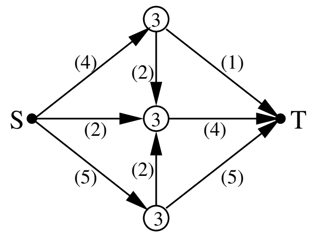
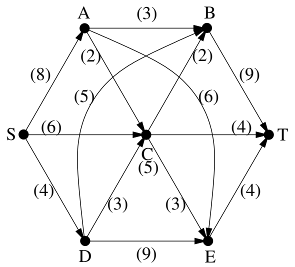
Combinatorics and Graph Theory 1.

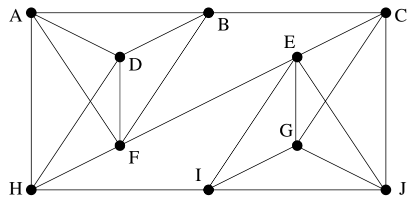
Exercise-set 11.

1. Determine the value of a maximum flow in the networks with edge- and vertex capacities below, and prove that they are maximal.





2. At most how many pairwise edge-disjoint and vertex-disjoint paths are there between the following points in the graph below:
a) B and I, b) A and J, c) B and H.



3. The graph G on 15 vertices is constructed from three cycles, on 4, 5 and 6 vertices each, in such a way that each vertex of the 5-vertex cycle was connected (with one edge) to all the other vertices of the other two cycles. Let s be a vertex of the 4-vertex cycle, and t be a vertex of the 6-vertex cycle.
a) At most how many pairwise vertex-disjoint paths are there in G between s and t?
b) At most how many pairwise edge-disjoint paths are there in G between s and t?

4. Determine the vertex- and edge connectivity numbers ($\kappa(G)$ and $\lambda(G)$) of the following graphs:
a) the graph consisting of the vertices and edges of a cube,
b) the complete bipartite graph $K_{m,n}$, where $m \geq n$.

5. The vertices of an 18-vertex graph G can be divided into 3 classes of six vertices each, in such a way that 2 vertices are adjacent if and only if they are in different classes. Determine the largest integer k for which G is k-vertex-connected ($\kappa(G)$), and the largest integer l for which G is l-edge-connected ($\lambda(G)$).

6. (MT+'19) We obtain the graph G by deleting the edges of a perfect matching from a complete graph on 10 vertices. Determine the vertex- and edge connectivity numbers ($\kappa(G)$ and $\lambda(G)$) for G.

7. (MT++'19) Let the vertices of the graph G be $v_1, v_2, \ldots, v_{20}$, and let the vertices v_i and v_j be adjacent if and only if $|i-j| = 1$ or 5. Determine the vertex- and edge connectivity numbers ($\kappa(G)$ and $\lambda(G)$) for G.

8. Show that a k-(vertex-)connected graph G on n vertices has at least $kn/2$ edges.

9. Prove that an $n/2$-(vertex-)connected graph on n vertices contains a Hamilton cycle.

10. (MT'19) Show that if G is a simple, undirected, 12-edge-connected graph on 20 vertices, then its complement cannot be 8-edge-connected.

11. We connect two disjoint complete graphs on 5 vertices with 3 edges, in such a way that the resulting graph G is simple. Is it true in all cases that G is
a) 3-(vertex-)connected;
b) 3-edge-connected?

12. At most how many edges can be deleted from the complete graph on 10 vertices in such a way that the remaining graph is 4-edge-connected?

13. Construct a simple graph which is 2-vertex-connected, 3-edge-connected and has minimum degree 4.

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14. Show that if a graph is 3-(vertex-)connected, then it contains a cycle of even length.

15. Let G be a 3-(vertex-)connected graph with 100 vertices and let $x, y \in V(G)$ be two different vertices. Show that there is a path from x to y whose length (i.e. the number of edges in it) is not greater than 33.

16. Prove that a graph G is 2-connected if and only if for every pair of vertices $x, y \in V(G)$ there is a cycle going through x and y.

17. The graph G contains a vertex from which 3 pairwise edge-disjoint paths go to any other vertex. Show that there are 3 pairwise edge-disjoint paths between any two vetices of G.

18. a) Let G be a k-connected graph, and G' be a graph obtained by adding a new vertex of degree at least k to G. Show that if G' is a simple graph, then it is k-(vertex-)connected as well.
b) Let G be a k-connected graph, and $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be two disjoint point sets in it. Prove that there are k (completely) vertex-disjoint paths in G connecting A and B.

19. Let G_1 and G_2 be two graphs on the same point set such that their edge-sets are disjoint, and let $G_1 + G_2$ be the graph whose point set is the common point set of the two graphs and the edge-set is the union of the two edge-sets.
a) Prove that $\lambda(G_1 + G_2) \geq \lambda(G_1) + \lambda(G_2)$.
b) Is it true that $\kappa(G_1 + G_2) \geq \kappa(G_1) + \kappa(G_2)$?