Combinatorics and Graph Theory 1.

Exercise-set 5.

1. In a company of 12 everybody knows at least 6 others (acquaintances are mutual). Show that this company can be seated around a round table in such a way that everybody knows his/her neighbors.
2. The simple graph G has $2k+1$ vertices. One of its vertices has degree k, and all the other vertices have degree at least $k+1$. Prove that G contains a Hamilton cycle.
3. In a company of 20 everybody knows the same number of people (acquaintances are mutual). Show that this company can be seated around a round table in such a way that either everybody knows his/her neighbors or nobody knows his or her neighbors.
4. There are 50 guests at a banquet, each of them knows at least 5 people from the others. (Acquaintances are mutual.) No matter how we choose 3 or 4 from the guests they cannot sit down to a round table in such a way that everybody knows both of his/her neighbors. Show that in this case all the guests can be seated around a round table for 50 persons in such a way that any two people who sit next to each other, but don't know each other have a common friend among the guests.
5. Show that for each $n \geq 5$ there is a graph G on n vertices such that both G and its complement contain a Hamilton cycle.
6. Give a simple, connected graph G on 8 vertices, whose complement is also connected, and neither G nor its complement contain a Hamilton cycle.
7. In the simple graph G on $2k+1$ vertices each vertex has degree at least k. Prove that G contains a Hamilton path.
8. In a simple graph on 20 vertices the degree of each vertex is at least 9. Prove that we can add one new edge to the graph in such a way that the resulting graph contains a Hamilton path.
9. In the simple graph G on 201 vertices the degree of each vertex, except for v, is at least 101. About v we only know that it is not an isolated vertex. Show that G contains a Hamilton path.
10. Show that if G is a simple 9-regular graph on 16 vertices, then we can delete 8 edges of G in such a way that the remaining graph contains an Euler circuit.
11. In a simple graph on 20 vertices the degree of each vertex is 8. Prove that we can add 20 new edges to the graph in such a way that the resulting graph is still simple and contains an Euler circuit.
12. Let G be a simple graph on $2k$ vertices in which the degree of each vertex is $k-1$, where $k>1$ is an integer. Prove that we can add k new edges to G in such a way that the resulting graph contains a Hamilton cycle.
13. Can we list all the four-element subsets of the set $\{1,2,\ldots,10\}$ in one sequence in such a way that the subsets which are next to each other in the sequence have at least two elements in common (and each subset appears exactly once in the sequence)?

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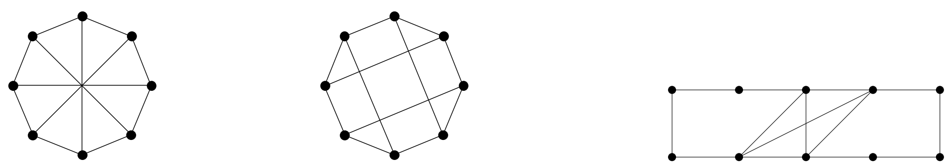
14. Determine whether the first two graphs below are bipartite or not:



15. At least how many edges must be deleted from the third graph above to get a bipartite graph?
16. 7 knights are put on a chessboard in such a way that each of them attacks at least two others. Show that there is such a knight among them which attacks three others.
17. Let the vertices of the graph G be the all the 0-1 sequences of length 5, and two sequences be adjacent if they differ in eactly one position. Is G a bipartite graph?
18. Is there a simple bipartite graph on at least 5 vertices whose complement is also a bipartite graph?
19. In a graph on 99 vertices two vertices have degree 3, and the degree of the other vertices is 4. Show that the graph contains an odd cycle.

20. Determine all the nonisomorphic simple graphs G on 8 vertices for which $\chi(G) = 2$ but if we add any edge to G (between two nonadjacent vertices) then for the graph G' obtained this way $\chi(G') = 3$ holds.

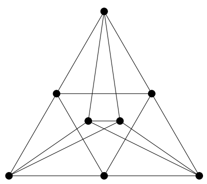
21. Determine all the nonisomorphic simple graphs G on n vertices for which $\chi(G) = 3$ but if we delete any vertex from G (together with the edges adjacent to it) then for the graph G' obtained $\chi(G') = 2$ holds.

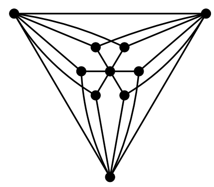
22. Determine the chromatic number of the graph of the regular octahedron. (The octahedron has 6 vertices and 8 triangular faces.)

23. Let the vertices of the graph G be the squares of the chessboard, and two vertices be adjacent if and only if the corresponding squares can be reached from each other by one move of a rook. Determine $\chi(G)$, the chromatic number of G. (A rook in chess can move either horizontally or vertically, and in one move it can go to any square along the selected line.)

24. Let the vertices of the graph G be the integers 1,2,...,100, and two vertices, m and n be adjacent if and only if $m + n$ is odd. Determine $\chi(G)$, the chromatic number of G.

25. Determine the chromatic number of the graphs below:





26. Let G be the graph obtained from a regular 11-sided polygon by adding all the shortest diagonals to it (i.e. G has 11 vertices and 22 edges). Determine $\chi(G)$ and $\omega(G)$.

27. Determine the chromatic number of the complement of the cycle on n vertices.

28. Let the vertices of the graph G be the numbers 1,2,...,2015, and two vertices be adjacent if and only if the difference of the corresponding numbers is at most 9. Determine $\chi(G)$, the chromatic number of G.

29. Let the vertex set of the graph G be $V(G) = \{1, 2, \dots, 30\}$. Let the vertices $x, y \in V(G)$ be adjacent in G if the difference of the numbers x and y is at least 7. Determine $\chi(G)$, the chromatic number of G.

30. Let the vertex set of the graph G be $V(G) = \{1, 2, \dots, 100\}$. Let the vertices $x, y \in V(G)$ be adjacent in G if $x \neq y$ and $100 \leq x \cdot y \leq 400$. Determine the value of $\chi(G)$.

31. Let the vertices of the graph G be the numbers 1,2,...,15, and two vertices be adjacent if and only if one the corresponding numbers divides the other. Determine $\chi(G)$, the chromatic number of G.

32. Let the vertices of the graph G be the numbers 1,2,...,30, and two vertices be adjacent if and only if one the corresponding numbers are relatively prime. Determine $\chi(G)$, the chromatic number of G.

33. In a simple graph G on 10 vertices the degree of each vertex is 8. Determine the chromatic number of G.

34. a) Prove that $|E(G)| \geq \binom{\chi(G)}{2}$ holds for every graph G.
b) Prove that in a coloring a graph G with $\chi(G)$ colors every color class contains a vertex v such that v has a neighbor in every other color class.

35. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs on the same vertex set, and let $G = (V, E_1 \cup E_2)$. Prove that $\chi(G) \leq \chi(G_1)\chi(G_2)$.

36. Let the vertex set of the graph G be $V(G) = \{1, 2, \dots, 2015\}$. Suppose that every vertex of G is adjacent to at most 10 smaller numbers. Prove that $\chi(G) \leq 11$.

37. In the simple graph G apart from 100 exceptional vertices the degree of each vertex is at most 99. Prove that $\chi(G) \leq 100$.

38. A simple graph G on 10 vertices contains one vertex of degree 5, one of degree 4, one of degree 3, and the rest of the vertices have degree 2. Show that G can be colored with 3 colors.

39. Prove that every simple planar graph can be colored with 6 colors (without using the four color theorem).