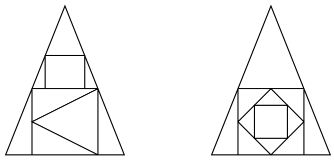
Combinatorics and Graph Theory 1.

Exercise-set 4.

1. If it is possible, draw the figures below with one line, without lifting the pen.



2. Let the vertices of the graph G be all the 2-element subsets of the 8-element set $S = \{a, b, c, d, e, f, g, h\}$, and two vertices be adjacent if and only if the corresponding subsets are disjoint. Does the graph G contain an Euler circuit?

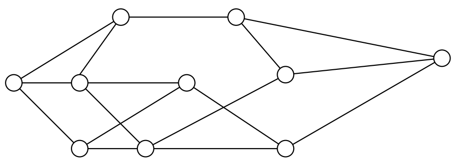
3. Let the vertices of the graph G be all the 0-1 sequences of length 4, and two vertices be adjacent if and only if the corresponding sequences differ in exactly two digits. Does this graph G contain an Euler circuit?

4. Let the vertices of the graph G_n be all the 0-1 sequences of length n, and two vertices be adjacent if and only if the corresponding sequences differ in exactly one digit. For which values of n does the graph G_n contain an Euler circuit?

5. 11 children play a game. They stand in a circle, and one of them starts passing a ball to somebody else, who in turn passes it on to a third child, etc. The rules are the following: nobody can throw the ball to somebody he/she has thrown it before, also nobody can throw the ball to somebody who has thrown it to him/her before, and nobody can throw the ball to either of the two children standing next to him/her in the circle. At most how many passes are possible in the game under these rules?

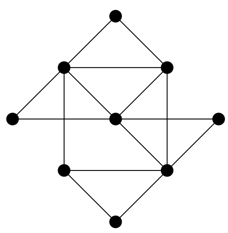
6. Show that there exists an integer n (in decimal form), in which the sum of the adjacent digits is never 9, but no matter how we choose two different integers between 0 and 9 whose sum is not 9, the two chosen numbers occur exactly once in n as adjacent digits (in some order).

7. The graph below is the sketch of a drainage system. At every junction where the drains meet a ladder leads up to the surface. Terrorists might have poisoned the network somewhere, so each drain needs to be disinfected. The method is that a technician in special clothing climbs though the drains. He cannot enter the already disinfected part again in order not to recontaminate it. At least how many times must he climb to the surface in order to finish the whole disinfection?



8. In a connected graph G $2k$ $(k \geq 1)$ vertices have odd degrees. Show that G is the union of k disjoint trails. Can it be the union of less than k trails?

9. At least how many edges need to be added to the graph below so that the resulting graph is still simple and contains an Euler trail?



10. The graph G on 10 vertices is constucted from two (vertex-disjoint) paths on 5 vertices in such a way that we connect each vertex of one path with every vertex of the other path. At most how many edges can a trail in G contain?

11. The graph G contains a circuit which contains every edge of G an odd number of times. Does it follow that G contains an Euler-circuit as well?

12. In a simple graph on 20 vertices the degree of each vertex is 6. Prove that we can add one new edge to the graph in such a way that the resulting graph is still simple and contains an Euler trail.

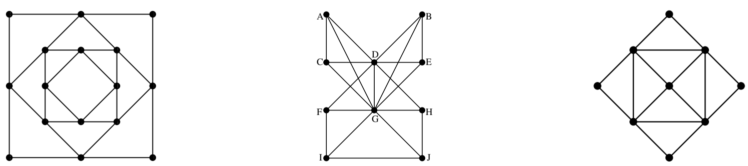
13. Prove that if all the degrees in the graph G are even, then G is the union of edge-disjoint cycles.

14. Prove that if the graph G contains an Euler circuit, then from every subset of vertices S there is an even number of edges going to the complement of S.

15. We know that the graph G contains an Euler circuit, and let e and f be two adjacent edges of G. Is it true that G contains an Euler circuit in which e and f follow each other?

16. Decide whether the statements below are true or not.
a) If we delete the edges of one cycle from the graph G and the remaining graph G' contains an Euler circuit, then G contains one as well.
b) If G is connected and we delete the edges of one cycle from G and the remaining graph G' contains an Euler circuit, then G contains one as well.
c) I G contains an Euler circuit and we delete the edges of one cycle from G then the remaining graph G' contains as well.

17. Do the following graphs contain a Hamilton cycle? And a Hamilton path?



18. Let the vertices of the graph G be the squares of a 5×5 chessboard, and two vertices be adjacent if and only if the corresponding squares have a common edge. The graph G_1 is obtained from G by deleting a vertex corresponding to one of the corners of the chessboard from it (so G_1 has 24 vertices). The graph G_2 is obtained from G by deleting two vertices corresponding to opposite corners of the chessboard from it (so G_2 has 23 vertices).
a) Does G_1 contain a Hamilton cycle? And a Hamilton path?
b) Does G_2 contain a Hamilton cycle? And a Hamilton path?

19. Let the vertex set of the graph G be $V(G) = \{1, 2, \ldots, 20\}$. Let the vertices $x, y \in V(G)$ be adjacent in G if $x \neq y$ and $x \cdot y$ is divisible by 3 or 5 (or both).
a) Does G contain a Hamilton path?
b) Does G contain a Hamilton cycle?

20. Let the vertices of the graph G_n be all the 0-1 sequences of length n, and two vertices be adjacent if and only if the corresponding sequences differ in exactly one digit. Does the graph G_n contain a Hamilton cycle?

21. At least how many edges must be added to the graphs below so that the graphs obtained contain a Hamilton cycle?

22. The graph G is a star on 101 vertices (i.e. G has one vertex of degree 100 and hundred vertices of degree 1). At least how many edges must be added to G so that the graph obtained contains a Hamilton cycle?

23. a) Show that it is impossible to visit each square of a 4×4 chessboard (exactly once) with a horse.
b) Show that it is impossible to visit each square of a 5×5 chessboard (exactly once) with a horse such that in the 25th move we arrive back to the starting square.

24. Let G be a connected graph and let C be a cycle in G, for which it holds that if we delete any edge from it then we get a longest path in G. Prove that in this case C is a Hamilton cycle of G.