Combinatorics and Graph Theory 1.

Exercise-set 3.

1. a) In a tree the degree of each vertex is 1 or 2 or 3. How many vertices of degree 1 are there if there are 5 vertices of degree 3?
   b) Draw two such trees in which the number of vertices of degree 2 are different.
2. Prove that in a tree on n vertices there cannot be exactly $n-3$ vertices of degree 2.
3. In a tree only two kinds of degrees occur, one of them 9 times, the other one 92 times. What are the two degrees?
4. Show that if all the degrees in a tree are odd, then the number of edges is also odd.
5. How many vertices does the tree T have if the number of its edges is exactly one-tenth of the number of edges of its complement?
6. Determine all the trees (on at least two vertices) which are isomorphic to their complement.
7. (*) Prove the following: if T_1 and T_2 are two trees on the same vertex set and e_1 is an edge of T_1 then there is an edge e_2 of T_2 such that both $T_1 - e_1 + e_2$ and $T_2 - e_2 + e_1$ are trees.

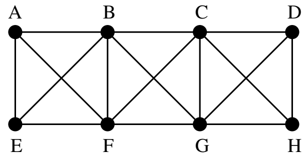
——

8. Let the vertex set of the simple graph be $V(G) = \{1, 2, \dots, 10\}$. Let the vertices $x, y \in V(G)$, $x \neq y$ be adjacent if and only if $|x - y| \leq 2$. Does G have a spanning tree, which
   a) contains all the edges $\{x, y\}$ of G for which $x, y \leq 3$ holds;
   b) contains all the edges $\{x, y\}$ of G for which $|x - y| = 2$ holds?
9. A simple connected graph on 100 vertices has 102 edges. Show that the graph contains three pairwise different cycles. (Two cycles are different if their edge sets are not the same.)
10. A simple connected graph on 100 vertices has 100 edges. Show that the graph contains three pairwise different spanning trees. (Two spanning trees are different if their edge sets are not the same.)
11. A graph on 20 vertices has 18 edges and 3 components. Show that exactly two of its components are trees.
12. Show that every connected graph contains a vertex whose deletion doesn't disconnect the graph.

——

13. a) Is there a plane graph in which the number of vertices, edges and regions are all divisible by 3?
    b) Is there a connected graph with the above properties?
14. Is there a simple connected plane graph which has half as many vertices as regions?
15. In a simple, connected plane graph the degree of each vertex is 4, and the number of the edges is 16. Determine the number of regions of the drawing.
16. A convex polyhedron has 20 vertices and 12 faces. Each face of the polyhedron is bounded by the same number of edges. What is this common number?
17. Each face of a convex polyhedron is a (not necessarily regular) quadrangle or an octogon. Furthermore we know that exactly 3 faces meet at each vertex of the polyhedron. What is the difference of the number of quadrangular and octogonal faces?
18. Let G be a simple, connected plane graph on $n \geq 3$ vertices, all of whose regions are triangles. Show that G has exactly $3n - 6$ edges. Draw such a graph.
19. a) Show that in a simple planar graph the minimum degree is at most 5.
    b) In the simple planar graph G the minimum degree is 5. Show that in this case G contains at least 12 vertices of degree 5.
    c) Is the statement true with 13 instead of 12?
20. a) Pove that if G is a simple planar graph on at least 11 vertices then at least one of G and its complement $\overline{G}$ is not planar.
    b) Give a simple graph G on 8 vertices such that both G and $\overline{G}$ are planar.

21. At most how many edges can be added to the graph below in such a way that we get a simple planar graph? (We add edges only between already existing vertices.)

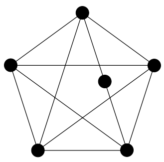
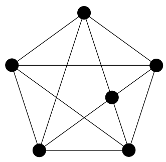
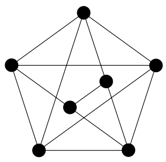
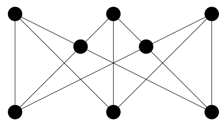


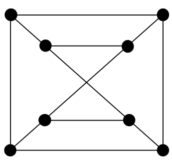
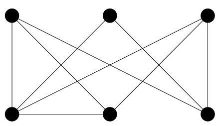
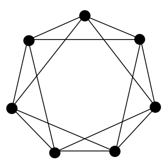
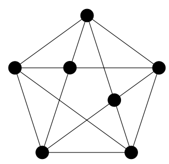
22. Suppose that $G = (V, E)$ is a simple graph whose set of edges E is the union of the disjoint sets of edges E_1, E_2 and E_3, where all the three subgraphs (V, E_1), (V, E_2) and (V, E_3) are spanning trees of G. Show that in this case G is not planar.

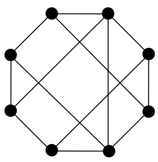
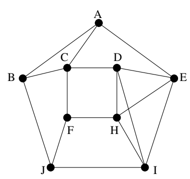
23. a) Show that in a drawing of K_8 in which three edges cannot go through a common point there are at least 10 edge-crossings.
b) At least how many edge-crossings are there in a drawing of $K_{4,4}$ if three edges cannot go through a common point?

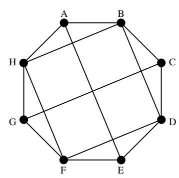
24. a) Determine the maximum of $n(G) + e(G) - 2r(G)$ over all connected simple plane graphs on 100 vertices.
b) Determine the maximum of $n(G) + 2e(G) - r(G)$ over all connected simple plane graphs on 100 vertices.

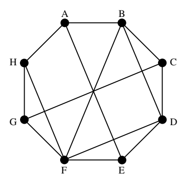
25. Decide whether the following graphs are planar or not:

a) 
b) 
c) 
d) 

e) 
f) 
g) 
h) 

i) 
j) 

k) 

l) 

26. In the graph G the degree of each vertex is at most 3. Furthermore we know that each cycle of G contains at most 5 edges. Show that G is a planar graph.

27. The graph G doesn't contain a subgraph homeomorphic to $K_{2,3}$ (the complete bipartite graph on 2+3 vertices). Does it follow that G is planar?

28. a) Let G be a simple graph on at most 6 vertices. Pove that at least one of G and its complement $\overline{G}$ is planar.
b) Give a simple graph G on 8 vertices such that neither G nor $\overline{G}$ is planar.