Combinatorics and Graph Theory 1.

Exercise-set 2.

1. Is there a simple or arbitrary graph with the following degree-sequences:
a) 1,2,2,3,3,3;
b) 1,1,2,2,3,4,4;
c) 1,3,3,4,5,6,6;
d) the degree of each vertex is different?

2. The graph G on 6 vertices can contain multiple edges, but no loops. We know that the degree of any two vertices of G are different. At least how many edges are there in G? (That is, for which integer k does it hold that there is a graph with this property with k edges, but not with less than k edges?)

3. Let G be a simple graph on n vertices ($n \geq 3$) with only one vertex of even degree. How many vertices of even degree are there in $\overline{G}$, the complement of G?

4. In the simple graph G on 20 vertices 10 vertices have degree at most 7, and the other 10 vertices have degree at least 16. How many edges are there in G?

5. Prove that for the degree-sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ of a simple graph $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$ holds for all $k \in \{1, 2, \ldots, n\}$.

6. How many non-isomorphic simple graphs are there on 4 vertices?

7. How many non-isomorphic simple graphs are there with 50 vertices and 1223 edges?

8. Sketch all the non-isomorphic trees on 3, 4, 5 and 6 vertices!

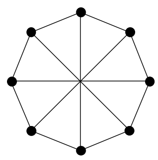
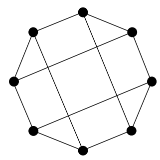
9. Draw all the pairwise non-isomorphic simple graphs with
a) $n = 5,\ e = 3$;
b) $n = 5,\ e = 7$;
c) $n = 5,\ e = 8$.
(Here n denotes the number of vertices, and e the number of edges.)

10. a) Is there a simple graph on 4 or 5 or 6 vertices which is isomorphic to its own complement?
b) Is there a 5-regular simple graph which is isomorphic to its own complement?

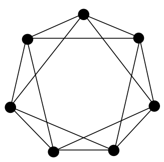
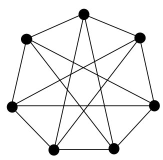
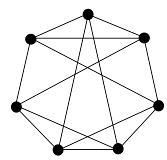
11. How many pairwise non-isomorphic connected simple graphs are there on 6 vertices which contain two vertices of degree 2 and four vertices of degree 3?

12. a) Determine the least number of vertices of a graph in which the length of the shortest cycle is exactly 4 and each of its vertices has degree 3.
b) How many non-isomorphic such graphs are there?

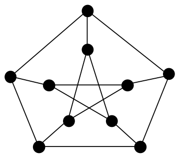
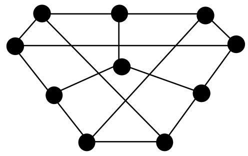
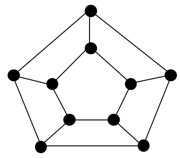
13. Let graph G consist of the vertices and edges of a cube. Which of the graphs below are isomorphic to G?

14. Which graphs are isomorphic from the three graphs below?

15. Let the vertices of the graph G be the 2-element subsets of the set $\{1, 2, 3, 4, 5\}$ and two vertices be adjacent if and only if the respective sets are disjoint. Which of the graphs on the next page are isomorphic to G?

16. We place 2 white and 2 black knights on a 3×3 chessboard in such a way that the knights of the same color stand in opposite corners. Can we achive with the usual moves in chess that the knights stand in opposite corners, but the opposite ones are of different color? (During the moves at most one knight can stand on a square.)

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17. a) In a graph on n vertices all the degrees are at least $\frac{n}{2}$. Does it follow that the graph is connected?
    b) And if we suppose that the graph is simple?

18. In a simple graph on 100 vertices each degree is at least 33. Show that we can add one edge to the graph in such a way that the resulting graph is connected.

19. In a simple graph on 23 vertices the degree of each vertex is at least 7. Show that no matter how we choose three vertices of the graph, there will be a path between two of them.

20. Show that for a simple graph G either G or its complement $\overline{G}$ is connected.

21. Let G be a simple graph and $v \in V(G)$ be a vertex of odd degree. Show that there is a path in G which starts at v and ends in a vertex of odd degree different from v.