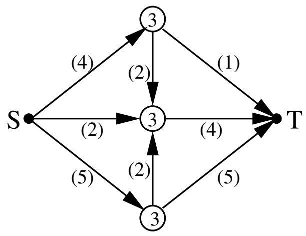
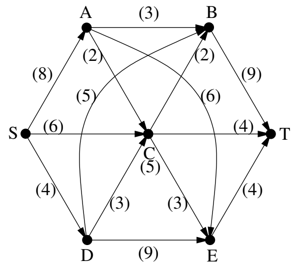
Exercise-set 10.

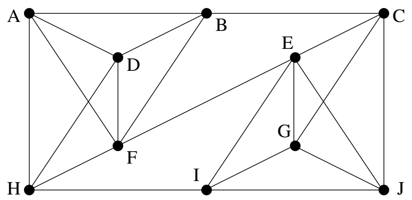
1. In a network all the capacities are integers. Which of the statements below holds always?
   a) Each maximum flow in the network has an integer value.
   b) There is a maximum flow in the network which takes an integer value on each edge.
   c) Each maximum flow in the network takes an integer value on each edge.
   d) What about the same questions if we substitute „integer” for „even number” everywhere?

2. Determine the value of a maximum flow in the networks with edge- and vertex capacities below, and prove that they are maximal.





3. At most how many pairwise edge-disjoint and vertex-disjoint paths are there between the following points in the graph below:
   a) B and I, b) A and J, c) B and H.



4. The graph G on 15 vertices is constructed from three cycles, on 4, 5 and 6 vertices each, in such a way that each vertex of the 5-vertex cycle was connected (with one edge) to all the other vertices of the other two cycles. Let s be a vertex of the 4-vertex cycle, and t be a vertex of the 6-vertex cycle.
   a) At most how many pairwise vertex-disjoint paths are there in G between s and t?
   b) At most how many pairwise edge-disjoint paths are there in G between s and t?

5. Determine the vertex- and edge connectivity numbers ($\kappa(G)$ and $\lambda(G)$) of the following graphs:
   a) the graph consisting of the vertices and edges of a cube,
   b) the complete bipartite graph $K_{m,n}$, where $m \geq n$.

6. The vertices of an 18-vertex graph G can be divided into 3 classes of six vertices each, in such a way that 2 vertices are adjacent if and only if they are in different classes. Determine the largest integer k for which G is k-vertex-connected ($\kappa(G)$), and the largest integer l for which G is l-edge-connected ($\lambda(G)$).

7. Show that a k-(vertex-)connected graph G on n vertices has at least $kn/2$ edges.

8. Prove that an $n/2$-(vertex-)connected graph on n vertices contains a Hamilton cycle.

9. We connect two disjoint complete graphs on 5 vertices with 3 edges, in such a way that the resulting graph G is simple. Is it true in all cases that G is
   a) 3-(vertex-)connected;
   b) 3-edge-connected?

10. At most how many edges can be deleted from the complete graph on 10 vertices in such a way that the remaining graph is 4-edge-connected?

11. Construct a simple graph which is 2-vertex-connected, 3-edge-connected and has minimum degree 4.

12. Show that if a graph is 3-(vertex-)connected, then it contains a cycle of even length.

13. Let G be a 3-(vertex-)connected graph with 100 vertices and let $x, y \in V(G)$ be two different vertices. Show that there is a path from x to y whose length (i.e. the number of edges in it) is not greater than 33.

14. Show that if $k \geq 1$ and G is an arbitrary k-edge-connected bipartite graph then by adding a new edge to G the new graph G' will either be bipartite or it will contain at least k odd cycles.

15. In the simple undirected graph G on n vertices the following holds: for every pair of nonadjacent vertices the sum of their degrees is at least $n + k - 2$ (where $k \geq 1$ is an integer). Show that G is k-(vertex-)connected.

16. Prove that a graph G is 2-connected if and only if for every pair of vertices $x, y \in V(G)$ there is a cycle going through x and y.

17. The graph G contains a vertex from which 3 pairwise edge-disjoint paths go to any other vertex. Show that there are 3 pairwise edge-disjoint paths between any two vetices of G.

18. a) Let G be a k-connected graph, and G' be a graph obtained by adding a new vertex of degree at least k to G. Show that if G' is a simple graph, then it is k-(vertex-)connected as well.
    b) Let G be a k-connected graph, and $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be two disjoint point sets in it. Prove that there are k (completely) vertex-disjoint paths in G connecting A and B.

19. Show that a 3-regular graph is k-edge-connected if and only if it is k-(vertex-)connected.

20. a) Let G_1 and G_2 be two graphs on the same point set such that their edge-sets are disjoint, and let $G_1 + G_2$ be the graph whose point set is the common point set of the two graphs and the edge-set is the union of the two edge-sets.
    a) Prove that $\lambda(G_1 + G_2) \geq \lambda(G_1) + \lambda(G_2)$.
    b) Is it true that $\kappa(G_1 + G_2) \geq \kappa(G_1) + \kappa(G_2)$?

21. At most how much can the edge- and vertex-connectivity of the union of two trees be?

22. Give efficient algorithms to determine the vertex and edge-connectivities of a given graph.