Rendszeroptimalizálás
2020. március 24.

Applications of Total Unimodularity on Bipartite Matching Problems – Part 1

As we have seen before, the incidence matrix of every directed graph is totally unimodular. However, the notion of an incidence matrix can also be defined for undirected graphs: incidence and non-incidence is simply denoted by 1 and 0, respectively.

Definition. *Assume that G is an undirected, loopless graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The* incidence matrix *$B(G)$ of G is an $n \times m$ matrix such that for every $1 \leq i \leq n$ and $1 \leq j \leq m$*

$$[B(G)]_{i,j} = \begin{cases} 1, & \text{if } e_j \text{ is incident to } v_i; \\ 0, & \text{if } e_j \text{ is not incident to } v_i. \end{cases}$$

The question presents itself if the incidence matrices of undirected graphs are also totally unimodular. The answer is no in general, as shown by the example of a triangle: it is easy to check that the determinant of its (3×3) incidence matrix is ± 2. However, the answer is yes if restricted to bipartite graphs.

Theorem. *The incidence matrix of every bipartite graph is totally unimodular.*

Proof. Let $G(A, B; E)$ be a bipartite graph (with A and B being the partition classes of the vertex set). Let H be the directed graph obtained from G by directing each edge from A towards B. (That is, if $\{a, b\} \in E(G)$ such that $a \in A$ and $b \in B$ then $(a, b) \in E(H)$.) Since H is a directed graph, its incidence matrix $B(H)$ is known to be totally unimodular. However, the incidence matrix $B(G)$ of G can be obtained from $B(H)$ by negating all of its rows that correspond to vertices in B. Since negating rows does not affect total unimodularity, $B(G)$ is also totally unimodular. $\square$

The Maximum Weight Bipartite Matching Problem

Previously we defined and gave an efficient algorithm for the Maximum Weight Bipartite Matching Problem: given a bipartite graph $G(A, B; E)$ and a weight function $w : E \to \mathbb{R}$ on its set of edges, a matching M is sought for which $\sum_{e \in E} w(e)$ is maximum possible. Now we consider this problem again in the context of integer programming with TU matrices.

The problem can very naturally be formulated as an integer program. A binary variable $x(e)$ is assigned to each edge e: $x(e) = 1$ and $x(e) = 0$ corresponds to $e \in M$ and $e \notin M$, respectively. The total weight of the chosen edges (that is, edges with $x(e) = 1$) is measured by the linear objective function $\sum_{e \in E} w(e)x(e)$. The fact that these edges form a matching easily translates to a set of linear inequalities: the sum of the variables $x(e)$ across all the edges e incident to a vertex v is restricted to be at most 1 for every vertex v. Finally, as usual, binarity of the variables can be guaranteed by the constraints $0 \leq x(e) \leq 1$, $x(e)$ integer. However, in this case the constraints $x(e) \leq 1$ are redundant, these can be dropped: if $x(e) \geq 2$ were true for an edge $e = \{a, b\}$ then the sum of the variables $x(e)$ across all the edges e incident to either a or b would exceed 1. All in all, we get the following IP formulation of the Maximum Weight Bipartite Matching Problem:

$$\begin{array}{lll} & \max : \sum_{e \in E} w(e)x(e) & \\ & \text{subject to} & \\ (1) & \forall v \in V(G) : \quad \sum\{x(e) : e \text{ is incident to } v\} & \leq 1 \\ (2) & \forall e \in E(G) : \quad x(e) & \geq 0 \\ (3) & \forall e \in E(G) : \quad x(e) & \text{integer} \end{array}$$

The matrix form of the above IP formulation is the following:

$$\max\{wx : Q \cdot x \leq \mathbf{1}, x \geq 0, x \text{ integer}\},$$

where the row vector w contains the edge weights $w(e)$, Q is the incidence matrix of G and $\mathbf{1}$ is a column vector with all its elements equal to 1. The coefficient matrix of the system of inequalities $Q \cdot x \leq \mathbf{1}, x \geq 0$ is $\begin{pmatrix} Q \\ -I \end{pmatrix}$ which is totally unimodular (by the above theorem and the previously proved simple lemma). Since the right hand sides are also integer, this implies (by the previously claimed theorem) that integrality constraints can be dropped, the optimum of the corresponding LP relaxation can be chosen to be integer. Consequently, the Maximum Weight Bipartite Matching Problem can be solved in polynomial time (simply by solving the LP relaxation).

Obviously, the efficient (even polynomial time) solvability of the Maximum Weight Bipartite Matching Problem was already known: this follows from Egerváry's algorithm (covered previously in this course). Moreover, Egerváry's algorithm yields an even more efficient solution to this problem than a general-purpose LP solver. The advantage of the latter approach is (similarly to the case of network flow problems seen before) that it can easily be adapted for more complex problems too. Furthermore, it has interesting and useful theoretical consequences – which will be the topic of the next class.