Extremal stable graphs

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Outline

Introduction

Question

Let Π be a graph property so that if $G_1 \in \Pi$ and G_1 is a subgraph of G_2, then $G_2 \in \Pi$. (i. e. being non-Π is a hereditary graph property.)

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- What is the miminum number of edges in hypo-hamiltonian graph?

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Examples:

- What is the minimum number of edges in a k-connected or k-edge connected graph?
- What is the miminum number of edges in hypo-hamiltonian graph?
- What is the mininum number of edges in graph that is still Hamiltonian after removing k edges (or vertices)?

Notations and definitions

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Definition (Stability)

Let H be a fixed graph. If the graph G has the property that removing any k edges of G, the resulting graph still contains (not necessarily spans) a subgraph isomorphic with H, then we say that G is k-stable with regard to H.

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By $S_H(k)$ we denote the minimum number of edges in any k-stable graph.

Note that there is no “in a graph with n vertices” in the definition.

A trivial case: $H = P_2$

Proposition

$S_{P_2}(k) = k + 1$

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For any graph H, we have $S_H(k) \geq k + |E(H)|$.

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Lemma

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Proof.

Otherwise there aren't enough edges to form H. □

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$S_{P_3}(k) = k + 2 = k + |E(P_3)|$

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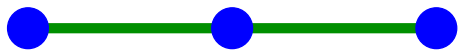

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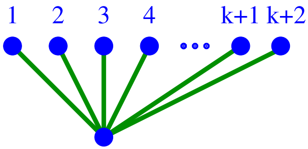
Extremal graph: A star with $k + 2$ edges.



An other easy case: $H = 2P_2$

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Extremal graph: $k + 2$ independent edges.

Linear bounds

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Let H be fixed.
(a) $S(k) \geq k + |E(H)|$.
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For a fixed H graphs we are interested in the exact value of $S(k)$ and also the extremal graphs.

Outline

Main result: $H = P_4$

Theorem

$S(1) = 4$, and for $k \geq 2$,

$$S(k) = k + \left\lceil \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil .$$

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The above formula is equivalent with the following:

Theorem

$S(1) = 4$, $S(2) = 6$, and if $k \geq 3$,

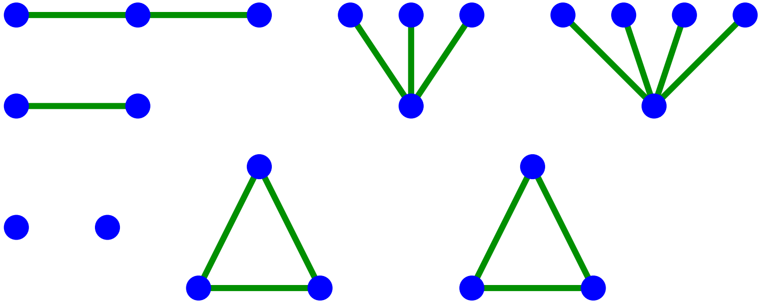
$$S(k) = \begin{cases} S(k-1) + 2 & \text{if } k = \binom{\ell}{2} \text{ for some integer } \ell \\ S(k-1) + 1 & \text{otherwise} \end{cases}$$

Outline

Covering with triangles and stars

Proposition

If G does not contain P_4 as a subgraph, then every component of G is a triangle or a star.



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That subgraph is a union of triangles and stars, and the number of stars is $n - (e - k)$.

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That subgraph is a union of triangles and stars, and the number of stars is $n-(e-k)$.
(In triangles, the number of edges is equal to the number of vertices, while in a star, the number of edges is 1 less, so we “lose” an edge for every star.) $\square$

Examples for the lower bound

We need to show that any graph with $< S(k)$ edges is not k-stable $\iff$ the vertices cannot be covered by $k + n - e$ stars and any number of triangles.

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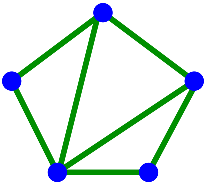

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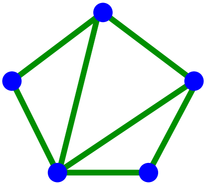

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$e = 7$

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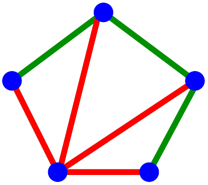

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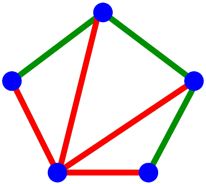

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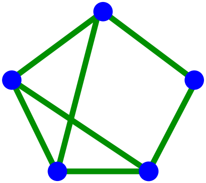

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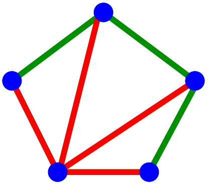

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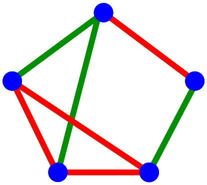

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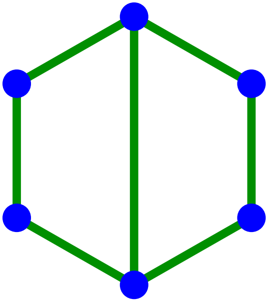

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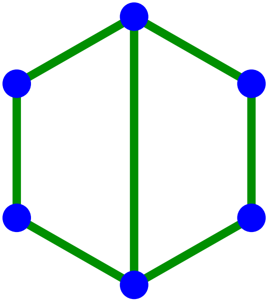

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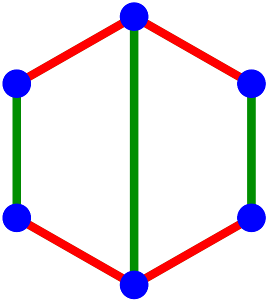

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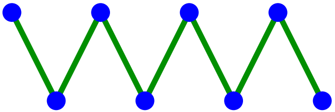

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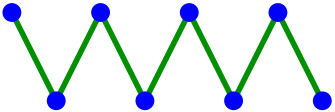

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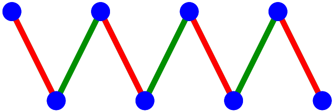

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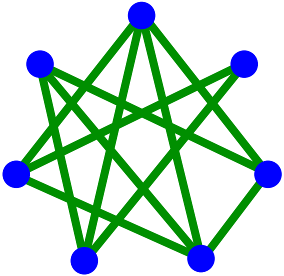

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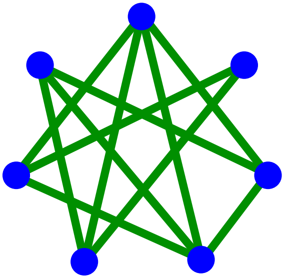

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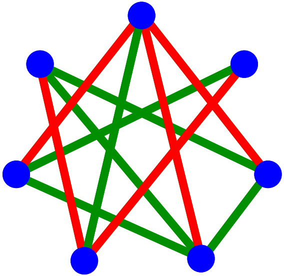

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Lower bound

Definition

Given a graph G with $e \geq 5$ edges on n vertices, let ℓ be the largest integer such that $e \geq \binom{\ell-1}{2} + 1$ (that is, ℓ is the smallest possible number of vertices that can fit e edges), and let $s = n - \ell$.
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If the graph G has $e \geq 5$ edges, then G can be covered by $s + 1$ stars and any number of triangles.

$S(k) \geq k + \left\lceil \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil$ follows directly from the above.

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The vertices can be covered by $s + 1$ stars.

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If only stars are used, then the centers of the stars forms a "dominating vertex set".

Lower bound - cases

The proof uses the following theorem:

Theorem (Vizing, 1965)

If G is a connected graph on n vertices and e edges, then the vertices can be dominated by a set of size

$$\beta(G) \leq \left\lfloor \frac{1 + 2n - \sqrt{8e+1}}{2} \right\rfloor$$

if $e \leq \frac{(n-2)(n-3)}{2}$.

Upper bound

Theorem (Upper bound)

$S(1) = 4$, *and for* $k \geq 2$, $S(k) \leq k + \left\lceil \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil$.

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Rephrased for coverings:

Theorem

For every $k \geq 2$, there exists a graph G with $e = k + \left\lceil \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil$ edges that is k-stable, that is, it cannot be covered by $s = k + n - e$ stars and any number of triangles.

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Proof.

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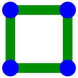
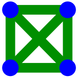
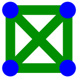
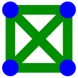
1. If $3 \nmid \ell$, then an almost complete graph,
2. If $3 | \ell$, then a complete graph with pendant edges.

$\square$

Constructions

k	$S(k)$	extremal graph
1	4	
1	6	
1	6	
1	6	

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Application

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A $v_1, v_2, \ldots, v_n$ permutation of the vertices of an r-regular hypergraph is a Hamiltonian chain if any r (cyclically) consecutive vertices form an edge.

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A hypergraph is k-edge-hamiltonian if it has the property that removing any k edges, the resulting hypergraph still contains a Hamiltonian chain.

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A hypergraph is k-edge-hamiltonian if it has the property that removing any k edges, the resulting hypergraph still contains a Hamiltonian chain.

Theorem (Frankl, Katona)

For every 3-regular k-edge-hamiltonian hypergraph with h edges on n vertices,

$$h \geq \frac{S(k)}{3} n.$$

Bibliography

P. FRANKL, G. Y. KATONA,
Extremal k-edge-hamiltonian hypergraphs,
Discrete Mathematics (2008) **308**, pp. 1415–1424

M. PAOLI, W. W. WONG, C. K. WONG,
Minimum k-Hamiltonian graphs. II.,
J. Graph Theory (1986) **10**, no. 1, pp. 79–95

D. RAUTENBACH,
A linear Vizing-like relation between the size and the domination number of a graph,
J. Graph Theory (1999) **31**, no. 4, pp. 297–302

V. G. VIZING,
An estimate of the external stability number of a graph,
Dokl. Akad. Nauk SSSR (1965) **52**, pp. 729–731