Extremal stable graphs

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Outline

1. Introduction
2. Main result
3. Outline of proof
4. Application
Let \( \Pi \) be a graph property so that if \( G_1 \in \Pi \) and \( G_1 \) is a subgraph of \( G_2 \), then \( G_2 \in \Pi \). (i.e. being non-\( \Pi \) is a hereditary graph property.)

What is the minimum number of edges in a graph \( G \in \Pi \) on \( n \) vertices if removing any \( k \) edges (or vertices) from the graph still preserves \( \Pi \)?

Examples:
- What is the minimum number of edges in a \( k \)-connected or \( k \)-edge connected graph?
- What is the minimum number of edges in a hypo-hamiltonian graph?
- What is the minimum number of edges in a graph that is still Hamiltonian after removing \( k \) edges (or vertices)?
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Notations and definitions

We concentrate on the problem where $\Pi$ is the property that $G$ contains a given fixed subgraph $H$.

Definition (Stability) Let $H$ be a fixed graph. If the graph $G$ has the property that removing any $k$ edges of $G$, the resulting graph still contains (not necessarily spans) a subgraph isomorphic with $H$, then we say that $G$ is $k$-stable with regard to $H$.

Definition By $S_{H}(k)$ we denote the minimum number of edges in any $k$-stable graph.

Note that there is no "in a graph with $n$ vertices" in the definition.
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A trivial case: $H = P_2$

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**Extremal graph:** Any graph with $k + 1$ edges.

**Lemma**

*For any graph $H$, we have $S_H(k) \geq k + |E(H)|$.***
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**Extremal graph:** Any graph with $k + 1$ edges.

**Lemma**

*For any graph $H$, we have $S_H(k) \geq k + |E(H)|$.*

**Proof.**

Otherwise there aren’t enough edges to form $H$. □
An easy case: $H = P_3$

Proposition

$S_{P_3}(k) = k + 2 = k + |E(P_3)|$
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Extremal graph: A star with $k + 2$ edges.
An other easy case: $H = 2P_2$

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\[ S_{2P_2}(k) = k + 2 = k + |E(2P_2)| \]

Extremal graph: $k + 2$ independent edges.
Proposition

Let $H$ be fixed.

(a) $S(k) \geq k + |E(H)|$.

(b) $S(k) \leq (|V(H)| + 1)k$ if $k$ is large enough.
Linear bounds

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(a) is trivial,

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For a fixed $H$ graphs we are interested in the exact value of $S(k)$ and also the extremal graphs.
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2. Main result
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Main result: $H = P_4$

**Theorem**

$S(1) = 4$, and for $k \geq 2$,

$$S(k) = k + \left\lceil \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil.$$
Main result: $H = P_4$

**Theorem**

$S(1) = 4$, and for $k \geq 2$,

$$S(k) = k + \left\lceil \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil.$$

The above formula is equivalent with the following:

**Theorem**

$S(1) = 4$, $S(2) = 6$, and if $k \geq 3$,

$$S(k) = \begin{cases} S(k - 1) + 2 & \text{if } k = \binom{\ell}{2} \text{ for some integer } \ell \\ S(k - 1) + 1 & \text{otherwise} \end{cases}.$$
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Covering with triangles and stars

**Proposition**

*If G does not contain \( P_4 \) as a subgraph, then every component of G is a triangle or a star.*
Proposition

If $G$ is a graph with $e$ edges on $n$ vertices, then
$G$ is $k$-stable $\iff$ the vertices of $G$ cannot be covered by $k + n - e$
stars and any number of triangles.

Proof.
$G$ is not $k$-stable if there is a subgraph with $e - k$ edges of $G$
such that it does not contain $P_4$.
That subgraph is a union of triangles and stars, and the number of
stars is $n - (e - k)$.
(In triangles, the number of edges is equal to the number of vertices,
while in a star, the number of edges is 1 less, so we “lose” an edge for
every star.)
Covering with triangles and stars

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Proposition

If $G$ is a graph with $e$ edges on $n$ vertices, then $G$ is $k$-stable $\iff$ the vertices of $G$ cannot be covered by $k + n - e$ stars and any number of triangles.

Proof.

$G$ is not $k$-stable if there is a subgraph with $e - k$ edges of $G$ such that it does not contain $P_4$. That subgraph is a union of triangles and stars, and the number of stars is $n - (e - k)$. (In triangles, the number of edges is equal to the number of vertices, while in a star, the number of edges is 1 less, so we “lose” an edge for every star.)
Examples for the lower bound

We need to show that any graph with $< S(k)$ edges is not $k$-stable $\iff$ the vertices cannot be covered by $k + n - e$ stars and any number of triangles.
Examples for the lower bound

We need to show that any graph with \(< S(k)\) edges is not \(k\)-stable
\[\iff\] the vertices cannot be covered by \(k + n - e\) stars and any number of triangles.

\(k = 3, S(k) = 8\)
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$n = 5$
$e = 7$
$k + n - e = 1$
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$k = 6, S(k) = 12$
Examples for the lower bound

We need to show that any graph with \(< S(k)\) edges is not \(k\)-stable \iff the vertices cannot be covered by \(k + n - e\) stars and any number of triangles.

\(k = 6, S(k) = 12\)

\(n = 7\)
\(e = 11\)
\(k + n - e = 2\)
Examples for the lower bound

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Lower bound

Definition

Given a graph $G$ with $e \geq 5$ edges on $n$ vertices, let $\ell$ be the largest integer such that $e \geq \binom{\ell-1}{2} + 1$ (that is, $\ell$ is the smallest possible number of vertices that can fit $e$ edges), and let $s = n - \ell$. $s \geq 0$ because of the definition of $\ell$; $s$ measures how ‘spread-out’ $G$ is.
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Theorem (Lower bound)

If the graph $G$ has $e \geq 5$ edges, then $G$ can be covered by $s + 1$ stars and any number of triangles.
Lower bound

**Definition**

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**Theorem (Lower bound)**

If the graph $G$ has $e \geq 5$ edges, then $G$ can be covered by $s + 1$ stars and any number of triangles.

$S(k) \geq k + \left[\sqrt{2k + \frac{9}{4} + \frac{3}{2}}\right]$ follows directly from the above.
Lower bound - cases

For different values of $s$, the methods are different.
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**Lemma ($s = 0$ or $s = 1$)**

*The vertices can be covered by $s + 1$ stars and *at most* 1 triangle.*
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The proof is long but elementary.
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No triangles are needed.
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The vertices can be covered by $s + 1$ stars.

No triangles are needed.

If only stars are used, then the centers of the stars forms a ”dominating vertex set”.

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The proof uses the following theorem:

**Theorem (Vizing, 1965)**

If $G$ is a connected graph on $n$ vertices and $e$ edges, then the vertices can be dominated by a set of size

$$\beta(G) \leq \left\lfloor \frac{1 + 2n - \sqrt{8e + 1}}{2} \right\rfloor$$

if $e \leq \frac{(n-2)(n-3)}{2}$. 
Theorem (Upper bound)

\[ S(1) = 4, \text{ and for } k \geq 2, S(k) \leq k + \left\lceil \sqrt{2k + \frac{9}{4}} + \frac{3}{2} \right\rceil. \]
Upper bound

**Theorem (Upper bound)**

\[ S(1) = 4, \text{ and for } k \geq 2, \ S(k) \leq k + \left\lceil \sqrt{2k + \frac{9}{4} + \frac{3}{2}} \right\rceil. \]

Rephrased for coverings:

**Theorem**

*For every* \( k \geq 2 \), *there exists a graph* \( G \) *with* \( e = k + \left\lceil \sqrt{2k + \frac{9}{4} + \frac{3}{2}} \right\rceil \) *edges that is* \( k \)-*stable, that is, it cannot be covered by* \( s = k + n - e \) *stars and any number of triangles.*
Upper bound

Proof.
Let $\ell$ be the unique integer for which \( \binom{\ell - 2}{2} \leq k \leq \binom{\ell - 1}{2} - 1 \).

There are 2 types of constructions:

1. If $3 \mid \ell$, then an almost complete graph,
2. If $3 \nmid \ell$, then a complete graph with pendant edges.
Proof.

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### Constructions

<table>
<thead>
<tr>
<th>$k$</th>
<th>$S(k)$</th>
<th>Extremal Graph</th>
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</tr>
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<td><img src="image" alt="Cross" /></td>
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A $v_1, v_2, \ldots, v_n$ permutation of the vertices of an $r$-regular hypergraph is a Hamiltonian chain if any $r$ (cyclically) consecutive vertices form an edge.

A hypergraph is $k$-edge-hamiltonian if it has the property that removing any $k$ edges, the resulting hypergraph still contains a Hamiltonian chain.
A \( v_1, v_2, \ldots, v_n \) permutation of the vertices of an \( r \)-regular hypergraph is a Hamiltonian chain if any \( r \) (cyclically) consecutive vertices form an edge.

A hypergraph is \( k \)-edge-hamiltonian if it has the property that removing any \( k \) edges, the resulting hypergraph still contains a Hamiltonian chain.

**Theorem (Frankl, Katona)**

For every 3-regular \( k \)-edge-hamiltonian hypergraph with \( h \) edges on \( n \) vertices,

\[
h \geq \frac{S(k)}{3} n.
\]
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