**ITC2 Repeat Midterm, May 25, 2020.**

**General Rules.** Disclaimer: Google translate has been used for this section. The solutions have been translated by Padmini Mukkamala.

The purpose of the scoring guide is to ensure that the dissertations are evaluated uniformly by the correctors. Therefore, the guide the main ideas for solving each task (at least one possible) and the marks assigned to them communicates sub-scores. The guide is not intended to detail the complete solution of the tasks description; the steps described can be considered as a sketch of a solution with a maximum score.

The sub-scores indicated in the guide only accrue to the solver if the related idea is included in the dissertation as a step towards a clear, clearly described and justified solution. Thus, for example, stating the definitions and items in the material without knowing how to apply them does not deserve any points (even if any of the facts described are indeed used in the solution). Deciding the score based on the points indicated in the guide in light of the above is under the grader's full remedial authority.

A partial score is awarded for each idea or partial solution from which, with a suitable addition, a flawless solution to the problem would have been obtained. If a solver starts several several substantially different solutions for a task, he can be assigned to at most one score. If all the solutions or parts of solutions described are correct or correct, then the solution initiative worth the most subpoints is evaluated. However, if amongst several solution attempts there is a correct solution but also an incorrect one (with a substantial error), and it is not clear from the dissertation which the solver considered as correct, then the solution with fewer points is evaluated (even if this score is 0).

The sub-scores in the guide can be further divided if necessary. A good solution other than that described in the guide is, of course, worth a maximum point. Theorems can be stated without proof, but only those discussed in class.

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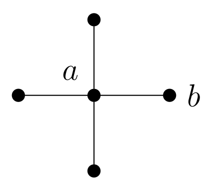
1. The statement is not true, but just stating that is awarded no points. A good counterexample with reasoning is awarded 10 points. Partial points are awarded in case of minor discrepancies, but not for solutions lacking a lot in reasoning, even if BFS is described completely (in general or on a concrete graph). There are numerous good counterexamples, the easiest being a star graph with the center being $a$ with degree at least 4, and $b$ one of the leaves (see pic). Here a BFS started at vertex $a$ picks $b$ as the fifth vertex visited. It is easy to see that a BFS started at $b$ must pick $a$ as the second vertex.

If the starting vertex of the BFS is counted as the 0th vertex, then also maximum points are awarded.



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2. A connected graph has an Eulerian walk if and only if the number of odd degree vertices is at most 2. (Both directions must be mentioned even though the solution uses only one). (1 pont)
When an edge is added to the graph (say, $G$), then two vertices have odd degree while the remaining six have even degree (1 pont)
so we must check if $G'$ (obtained by adding an edge) is connected and simple (it is enough since there are no isolated points). (1 pont)
Since degree of every vertex in $G$ is even and there are no isolated vertices, so the minimum degree is 2, so because the graph is simple, each component must have at least 3 vertices (points are not deducted if its not mentioned that the graph is simple), (2 pont)
so the graph has at most 2 components. (1 pont)
On the other hand, if $G$ is connected, then $G'$ is also connected (1 pont)
and we must only check that $G'$ is simple, and this is true because $G$ cannot be the complete graph because then the degree of each vertex is 7. (1 pont)
If $G$ has two components, we just draw an edge connecting the two, which will give us a connected graph with at most two vertices with odd degree, as required (1 pont)
and it will still be simple. (1 pont)

For a different solution: degree analysis: 1 point, to see that we need connectivity 1 point and why 1 point. The remaining 7 points for showing that a new edge can be added keeping the graph simple and making it connected.

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3. Let the vertices of the 3-cycle be $v, a, b$, and of the 7-cycle $v, c, d, e, f, g, h$. Then $va$, $vb$, $ab$ edges are not in the complement, (1 pont)
so $v, a, b$ can get the same color. (1 pont)
Since $cd$, $ef$, $gh$ edges are also not in the complement, (1 pont)
the vertices $c, d$ can get the second color, $e, f$ the third and $g, h$ the fourth.

(1 pont)
So the chromatic number is at most 4. (1 pont)
The vertices $a, c, e, g$ form a 4 clique in the complement of $G$, (4 pont)
so the chromatic number is at least 4 So, the chromatic number is exactly 4. (1 pont)

So a good coloring is worth 4 points, while finding a clique on 4 vertices is worth 4 points. The remaining 2 points are for the corresponding inequalities and combining these to derive the chromatic number. No points are to be awarded for a weaker bound on chromatic number.

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4. First solution: Let $L$ be a vertex cover of the graph (not necessarily minimal). If $L$ is $V(G)$, then it has 20 vertices. If not, then there is a vertex $v$ not in $L$. (2 pont)
Then, all the neighbors of $v$ must be in $L$ since it is a vertex cover, (5 pont)
and since the minimum degree is at least 10, $L$ must have at least 10 vertices in it. (2 pont)

Since $L$ was an arbitrary vertex cover, the minimum vertex cover must also have 10 vertices. (1 pont)

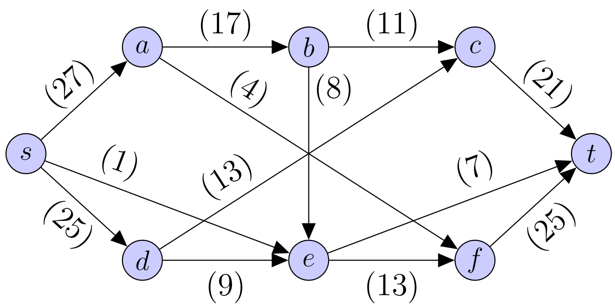
1 point must be deducted for not considering the $|L| = 20$ case.

Second solution: From Gallai's theorem $\alpha(G) + \tau(G) = n$, (1 pont)
so its enough to show that $\alpha(G)$ is at most 10. (1 pont)
If there was an independent vertex set $F$ of size 11 or more, then since the vertices in $F$ are not connected to each other, (4 pont)
and since $F$ is simple, (1 pont)
so each vertex in $F$ will have degree at most 9. (3 pont)

Third solution: By Dirac's theorem the graph has a Hamiltonian cycle, (1 pont)
because the degree of every vertex is at least $\frac{n}{2}$, (1 pont)
and it is a simple graph. (1 pont)
Since the graph has even number of edges, we can get a perfect matching of the graph by picking alternating edges of the Hamiltonian cycle, (3 pont)
which has 10 edges. (1 pont)
We know that $\nu(G) \leq \tau(G)$, (2 pont)
so the minimal vertex cover has at least 10 vertices. (1 pont)

No points are awarded for estimates which do not indicate or lead to a solution.

5.

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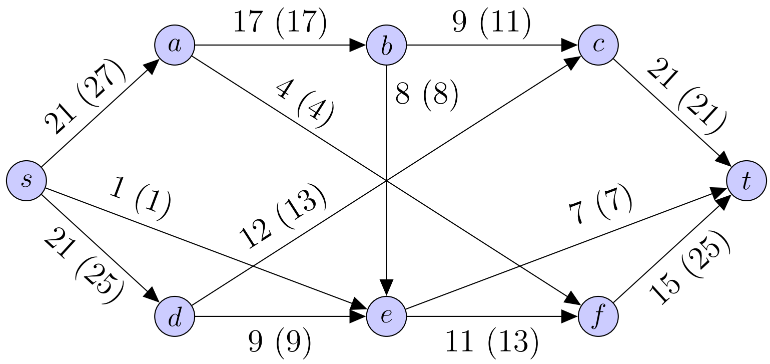
The cut $\{s, a, b, c, d\}$ has capacity 43, (2 pont)
it is the total of the capacities of $se$, $af$, $be$, $ct$, $de$ edges. (2 pont)
A flow of 43 can be seen in the following figure. (2 pont)
This can be deduced from the flow coming from $s$ (and since no flow enters it) (1 pont)




Since the capacity of any cut is greater than any flow, (2 pont)
so the capacity of any cut must be at least 43, so, $\{s, a, b, c, d\}$ is a minimal cut. (1 pont)

The last 3 points are for giving reasons for why the cut is minimal. The resoning may be different from the above mentioned, but no points are awarded for just mentioning Ford-Folkerson as reason.

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6. We will prove by contradiction that the graph is planar. Let us assume that the graph is non-planar. (0 pont)
So by the Kuratowski's theorem, it must have a subgraph $H$ topologically isomorphic to $K_{3,3}$ or $K_5$. (1 pont)
So,$H$ has a cycle, (1 pont)
deleting one edge of the cycle will not disconnect the graph. (2 pont)
Call these graphs with one less edge $G_1$ and $H_1$ respectively. Note that $H_1$ is topologically isomorphic to $K_{3,3}$ with one less edge or to $K_5$ with one less edge. (2 pont)
So $H_1$ also has a cycle, (1 pont)
from which we can delete an edge so that the new graph obtained from $G_1$ (lets call it $G_2$) is again connected. (1 pont)
This process can be repeated twice more ($K_{3,3}$ has 4 more, and $K_5$ has 6 more edges than their spanning trees). (1 pont)
After this process is repeated 4 times, the graph then obtained, $G_4$, has 20 vertices, and 18 edges, and is connected which is clearly a contradiction. (1 pont)

For different solutions:

We can remove 3 edges from $G$ to get a cycle free graph: 2 points. Reasoning: 1 point. This is not true for Kuratowski graphs: 2 points, reasoning: 1 point. This is further not true for graphs that are topologically isomorphic to the Kuratowski graphs: 2 points, reasoning: 1 point. So the graph is planar: 1 point.