Planar Graphs

Definition: The graph G is *planar* if it can be drawn on the plane without crossing edges.
The graph G is *plane* if it is drawn on the plane without crossing edges (i.e. it is a good drawing of a planar graph).

Definition: The *regions* of the plane graph G are the minimal parts of the plane enclosed by the edges of G (including the infinite region).

Proposition: G is planar if and only if G can be drawn on the sphere without crossing edges.

Corollary 1: Any region of G can be the outer (infinite) region.

Corollary 2: Graphs of convex polyhedra are planar.

Theorem (Euler's formula): If G is a connected plane graph on n vertices, with e edges and r regions, then $n - e + r = 2$.

Remark: If the plane graph G has k components, then $n - e + r = k + 1$.

Corollary 1: If G is a simple planar graph, then $e \leq 3n - 6$.

Corollary 2: If G is a planar graph in which each cycle has length at least 4, then $e \leq 2n - 4$.

Corollary 3: K_5 and $K_{3,3}$ are not planar graphs.

Definition: 1. the *subdivision* of an edge $\{u, v\}$ means replacing the edge by the edges $\{u, w\}$ and $\{w, v\}$, where w is a new vertex.
2. the graphs G and G' are *homeomorphic*, if one can be obtained from the other by a sequence of subdivisions.

Theorem (Kuratowski): G is a planar graph if and only if it contains no subgraph homeomorphic to either K_5 or $K_{3,3}$.

Theorem (Fary-Wagner): If G is a simple planar graph then it has a plane drawing in which all the edges are line segments.

Definition: The *dual*, G^* of a plane graph G is obtained by placing one vertex in each region of G and for each edge of G connecting the vertices of G^* which they separate by an edge.

Remarks: 1. G^* is a plane graph, $n^* = r$, $e^* = e$, and $r^* = n$ if G is connected.
2. G^* depends on the plane drawing of G, therefore a planar graph G can have non-isomorphic duals.
3. $(G^*)^*$ is not necessarily isomorphic to G.
4. Graphs of convex polyhedra have a unique dual.