Exercise-set 8. Solutions

1. $\Delta(G) = 4 \implies \chi_e(G) \geq 4$, and the edges of $G$ can be colored with 4 colors $\implies \chi_e(G) \leq 4$.

2. a) Vertices of the same color are independent.
   b) Edges of the same color are independent.

3. $\chi_e(K_5) \geq e/\nu = 10/2 = 5$ and $\chi_e(K_5) \leq \Delta(K_5) + 1 = 5$, so $\chi_e(K_5) = 5$.
   $\chi_e(K_6) \geq \chi_e(K_5) = 5$, and the edges of $K_6$ can be colored with 5 colors $\implies \chi_e(K_6) \leq 5$.
   (In general, $\chi_e(K_{2n+1}) = 2n + 1$ and $\chi_e(K_{2n}) = 2n - 1$.)

4. $\chi_e(K_{20}) = 19$ (ex. 2.), and a round corresponds to edges of the same color.

5. $\chi_e(G) \geq \chi_e(K_5) = 5$, and and the edges of $G$ can be colored with 5 colors $\implies \chi_e(G) \leq 5$.

6. a) $\chi_e(G) \geq e/\nu = 15/2 > 7$ and the edges of $G$ can be colored with 8 colors $\implies \chi_e(G) = 8$.
   b) $\chi_e(G) \geq e/\nu = 15/2 > 7$ and the edges of $G$ can be colored with 8 colors $\implies \chi_e(G) = 8$.

7. $|E(G)| = 1999 \cdot 10/2 = 9995$, $\nu(G) \leq 1999/2 = 999 \implies \chi_e(G) \geq 9995/999 > 10$ and $\chi_e(G) \leq \Delta(G) + 1 = 11 \implies \chi_e(G) = 11$.

8. $\chi_e(G) \geq e/\nu = (2k \cdot 3 + 2)/2k > 3$ (since $|V(G)|$ is odd) and $\chi_e(G) \leq \Delta(G) + 1 = 4 \implies \chi_e(G) = 4$.

9. For a $k$-regular graph on 9 vertices $\chi_e(G) = k + 1$, and $\overline{G}$ is 8 - $k$-regular $\implies \chi_e(G) \geq 9 - k$.

10. Any color class of edges forms a perfect matching (covers all the vertices).

11. $\nu(G) \geq e/\chi_e \geq 16/5 > 3$ (since $\chi_e(G) \leq \Delta(G) + 1 = 4$), and $\nu(G) \leq 9/2$.

12. a) The edges of $G$ can be colored with 3 colors; 2 colors for the Hamilton cycle (since it has an even length, because $|E| = 3|V|/2$ is an integer), and one for the remaining edges.
   b) The edges of $G$ cannot be colored with 3 colors.

13. $\chi_e(G), \chi_e(G - v) \in \{8, 9\}$.
    If $\chi_e(G - v) = 9 \implies \chi_e(G) = 9$.
    If $\chi_e(G - v) = 8$, then an 8-coloring of the edges of $G - v$ can be extended to an 8-coloring of the edges of $G$.

14. If we delete the edge $\{a, b\}$ then by exercise 8, $\chi_e(G) = 4$.

15. Yes, if we delete a matching (see exercise 3.).

16. The edges of the original graph can be colored with $\chi_e(G') + 2$ colors.

17. Yes, $K_5 \setminus \{\text{one edge}\}$ is like that.

18. $\chi_e(G) \geq 4$ and and the edges of $G$ can be colored with 4 colors $\implies \chi_e(G) \leq 4$.

19. $G = \{\text{rows, columns; selected squares}\}$ is a 3-regular bipartite graph $\implies \chi_e(G) = 3$.

20. $G$ is bipartite $\implies \chi_e(G) = \Delta(G) = 6$; or give a concrete edge-coloring.

21. $G$ is bipartite $\implies \chi_e(G) = \Delta(G) = 3$.

22. The other degree is 3, and trees are bipartite $\implies \chi_e(G) = \Delta(G) = 3$.

23. $G = \{\text{two vertex-disjoint cycles (which are bipartite)}\}$ and a 100-regular bipartite graph $\implies \chi_e(G) = 2 + 100 = 102$.

24. $\chi_e(G) = \Delta(G) = 9$

25. There are 36 minimum weight spanning trees of weight 19.

26. There are 999 minimum weight spanning trees of weight $2 + 3 + \cdots + 100 = 5049$.

27. The weight of a minimum weight spanning tree is 150.

28. One of them is a path from 1 to 10 of weight 18.

29. By Kruskal’s algorithm: when we get to $e$, we cannot create a cycle.

30. By Kruskal’s algorithm: the other edges of $C$ can be selected before $e$.

31. There must be a cut in $G$ consisting of edges of weight 100 only.

32. Order the edges of $G$ such that the edges of the given spanning tree come first among the edges of the same weight.