1. Construct a graph $G$: $V(G) =$ people, and the edges are the acquaintances. Then $\deg(v) \geq 6 = 12/2$ $\implies$ by Dirac’s theorem $\exists$ a Hamilton cycle.

2. The condition in Ore’s theorem holds for $G \implies \exists$ a Hamilton cycle.

3. Construct a graph $G$: $V(G) =$ people, and the edges are the acquaintances. $G$ is $k$-regular for some $k$. If $k \geq 10 \implies G$ contains a Hamilton cycle, if $k \leq 9 \implies \overline{G}$ contains a Hamilton cycle.

4. Construct a graph $G$: $V(G) =$ people, and the edges are the acquaintances. We get $G'$ by adding the edges between the second neighbors to $G$. In $G'$ the degree of each vertex is at least $5 + 5 \cdot 4 = 25 \implies G'$ contains a Hamilton cycle.

5. a) A cycle on $n$ vertices is like that (check).
   b) E.g. $K_7$ with the edge $\{u, v\}$ missing and the 8th vertex is connected to $u$.

6. Add a new vertex to $G$, and connect it to all the old vertices. Then the new graph contains a Hamilton cycle from which we can get a Hamilton path of $G$.

7. Delete $v$ from $G$. Then the new graph contains a Hamilton cycle from which we can get a Hamilton path of $G$.

8. The 8 edges must have pairwise no common endpoints (i.e. be independent). Every second edge of a Hamilton cycle will do (which exists because $\deg(v) \geq n/2 \forall v$).

9. We can add the edges of a Hamilton cycle of $\overline{G}$ (which exists by Dirac’s theorem).

10. Need to add $k$ pairwise non-adjacent edges (from $\overline{G}$). $\overline{G}$ contains a Hamilton cycle ($\deg_{\overline{G}}(v) = k, \forall v \in V(G)$). Every second edge of it will do.

11. The first graph is not bipartite (contains 5-cycles), but the second graph is.

12. Deleting 2 edges are enough, but less is not, since $\exists$ 2 edge-disjoint odd cycles in $G$.

13. The graph determined by the knights and attacks is bipartite (the two classes are to the white and black squares), and each of its degrees is at least $2 \implies \exists$ a degree $\geq 3$.

14. Yes (the two classes of vertices are sequences with an even or odd number of 1’s, resp.).

15. No (the complement contains a triangle).

16. The vertices cannot be divided into two classes (count the degrees).

17. Complete bipartite graphs are like that.

18. The graphs are exactly the odd cycles (so in particular $n$ must be odd). $G$ must contain an odd cycle (otherwise $\chi(G') = 2$), and cannot contain more vertices or edges.

19. $\omega(G) = 3 \implies \chi(G) \geq 3$, and $G$ can be colored with 3 colors $\implies \chi(G) \leq 3$, so $\chi(G) = 3$.

20. $\omega(G) = 8$ (each row and column is a clique) $\implies \chi(G) \geq 8$, and $G$ can be colored with 8 colors (colors are diagonal) $\implies \chi(G) \leq 8$, so $\chi(G) = 8$.

21. $G$ is bipartite (the two classes of vertices are the even and odd numbers, resp.) $\implies \chi(G) = 2$.

22. $\omega(G) = 4 \implies \chi(G) \geq 4$, and $G$ can be colored with 4 colors $\implies \chi(G) \leq 4$.

23. a), b) $\omega(G) = 3 \implies \chi(G) \geq 3$, but $G$ cannot be colored with 3 colors (proof!) $\implies \chi(G) \geq 4$. $G$ can be colored with 4 colors $\implies \chi(G) \leq 4$.

24. $\omega(G) = 3 \implies \chi(G) \geq 3$, but $G$ cannot be colored with 3 colors (proof!) $\implies \chi(G) \geq 4$. $G$ can be colored with 4 colors $\implies \chi(G) \leq 4$.

25. $\chi(G) \geq \lceil n/2 \rceil$ (at most 2 vertices can get the same color), and $G$ can be colored with this many colors $\implies \chi(G) = \lceil n/2 \rceil$. 

Exercise-set 5.+6.

Solutions
26. \(\omega(G) = 10\) (any 10 consecutive numbers form a clique) \(\implies \chi(G) \geq 10\), and \(G\) can be colored with 10 colors (periodically) \(\implies \chi(G) \leq 10\).

27. \(\omega(G) = 5\) (\(\{1, 8, 15, 22, 29\}\) is a clique) \(\implies \chi(G) \geq 5\), and \(G\) can be colored with 5 colors \(\implies \chi(G) \leq 5\).

28. \(\omega(G) = 11\) (\(\{10, 11, \ldots, 20\}\) is a clique) \(\implies \chi(G) \geq 11\), and \(G\) can be colored with 11 colors \(\implies \chi(G) \leq 11\).

29. \(\omega(G) = 4\) (the powers of 2 form a clique) \(\implies \chi(G) \geq 4\), and \(G\) can be colored with 4 colors (using the same color between consecutive powers of 2) \(\implies \chi(G) \leq 4\).

30. \(\omega(G) = 11\) (prime numbers and 1 form a clique) \(\implies \chi(G) \geq 11\), and \(G\) can be colored with 11 colors \(\implies \chi(G) \leq 11\).

31. \(\omega(G) = 5\) \(\implies \chi(G) \geq 5\), and \(G\) can be colored with 5 colors \(\implies \chi(G) \leq 5\).

32. \(G\) is \(K_{10}\) with a perfect matching deleted. \(\omega(G) = 5\) \(\implies \chi(G) \geq 5\), and \(G\) can be colored with 5 colors \(\implies \chi(G) \leq 5\).

33. \(\omega(G) = 6\) (\(\{1, 4, 7, 8, 9, 10\}\) is a clique) \(\implies \chi(G) \geq 6\), and \(G\) can be colored with 6 colors \(\implies \chi(G) \leq 6\).

34. YES. See exercise 22.

35. \(\omega(G) = 50\) (even numbers form a clique) \(\implies \chi(G) \geq 50\), and \(G\) can be colored with 50 colors \(\implies \chi(G) \leq 50\).

36. \(\omega(G) = 4\) \(\implies \chi(G) \geq 4\), and \(G\) can be colored with 4 colors \(\implies \chi(G) \leq 4\).

37. \(\chi(G) = 4\). See exercise 25.

38. Use the greedy coloring in the original (increasing) order of the vertices.

39. Order the vertices: first the exceptional ones, then the rest, and use the greedy coloring.

40. Use the greedy coloring in the decreasing order of the degrees.