Menger's Theorems

Theorem (Menger 1): For (any) two vertices s, t in a directed graph G, the maximum number of edge-disjoint directed paths from s to t is equal to the minimum number of edges covering all the directed paths from s to t.

Theorem (Menger 2): For (any) two vertices s, t in a directed graph G for which (s, t) is not and edge, the maximum number of vertex-disjoint directed paths from s to t is equal to the minimum number of vertices (without s and t) covering all the directed paths from s to t.

Theorem (Menger 3): For (any) two vertices s, t in a(n undirected) graph G, the maximum number of edge-disjoint paths between s and t is equal to the minimum number of edges covering all the $s - t$ paths.

Theorem (Menger 4): For (any) two vertices s, t in a(n undirected) graph G for which $\{s, t\}$ is not and edge, the maximum number of vertex-disjoint paths between s and t is equal to the minimum number of vertices (without s and t) covering all the $s - t$ paths.

Higher Connectivity of Graphs

Definition: A graph G is *k-vertex-connected* (or *k-connected* for short) if no matter how we delete $k - 1$ of its vertices, it remains connected, and it has at least $k + 1$ vertices.

Definition: The *vertex connectivity number*, $\kappa(G)$ of a graph G is the largest integer k for which G is k-vertex-connected.
In other words, $\kappa(G) = k$ if and only if the deletion of any set of $k - 1$ vertices doesn't disconnect G, but there is a set of k vertices whose deletion disconnects G.

Definition: A graph G is *l-edge-connected* if no matter how we delete $l - 1$ of its edges, it remains connected.

Definition: The *edge connectivity number*, $\lambda(G)$ of a graph G is the largest integer l for which G is l-edge-connected.
In other words, $\lambda(G) = l$ if and only if the deletion of any set of $l - 1$ edges doesn't disconnect G, but there is a set of l edges whose deletion disconnects G.

Remarks: 1. A graph is 1-vertex-connected = 1-edge-connected = connected in the old sense.
2. $\kappa(G), \lambda(G) \leq \min \deg(G)$.

Theorem (Menger 5): A graph is k-vertex-connected if and only if there are at least k vertex-disjoint paths between any two of its vertices and it has at least $k + 1$ vertices.

Theorem (Menger 6): A graph is l-edge-connected if and only if there are at least l edge-disjoint paths between any two of its vertices.

Corollary: If a graph is k-vertex-connected then it is also k-edge-connected, i.e. $\kappa(G) \leq \lambda(G)$ for every graph G.