Matchings

Definition: A *matching* in a graph G is an independent set of edges, i.e. a set of edges which share no endpoints.

Definition: The *matching number* of the graph G, $\nu(G)$ is the size of a maximum matching in G.

Remarks: 1. $\nu(G) \leq |V(G)|/2$ in any graph.
2. In a bipartite graph $\nu(G) \leq \min\{|A|, |B|\}$.

Definition: A matching of the graph G is *perfect*, if it covers all the vertices of G (i.e. $\nu(G) = |V(G)|/2$).

Theorem (Hall): A bipartite graph G has a matching covering A if and only if $|N(X)| \geq |X|$ for every subset X of A, where $N(X)$ denotes the set of neighbors of X (in B).

Corollary (Frobenius' theorem): A bipartite graph G has a perfect matching if and only if $|A| = |B|$ and $|N(X)| \geq |X|$ for every subset X of A.

Definition: A *covering set of vertices* in a graph G is a set of vertices which cover all the edges, i.e. at least one endpoint of each edge belongs to this set.

Definition: The *covering number* of the graph G, $\tau(G)$ is the size of a minimum covering set of vertices in G.

Proposition: $\nu(G) \leq \tau(G)$ in any graph G.

Proposition: $\tau(G) \leq 2\nu(G)$ in graph G without loops.

Theorem (König): In a bipartite graph G, $\nu(G) = \tau(G)$.

Definition: An *alternating path* in a graph G with respect to a matching M is a path whose every second edge belongs to M.

Definition: An *augmenting path* in a graph G with respect to a matching M is an alternating path whose endpoints are not covered by M.

By (the proof of) König's theorem, the following **algorithm** can be used to find a maximum matching in a bipartite graph:
1. select independent edges, as long as we can.
2. use augmenting paths to get larger matchings.
3. if there are no more augmenting paths, then the matching is maximum, and we can prove it by finding a covering set of vertices of the same size.

Theorem: A matching in (any) graph G is maximum if and only if there is no increasing path to it.

Theorem (Tutte): A graph G has a perfect matching if and only if the number of components with an odd number of vertices in $G \setminus X$ is at most $|X|$ for every subset X of $V(G)$.

Other graph parameters

Definition: The *independence number* of a graph G, $\alpha(G)$, is the size of the largest independent set of vertices in G.

Definition: A *covering set of edges* in a graph G is a set of edges which cover all the vertices, i.e. the set of all their endpoints is $V(G)$.

Definition: $\rho(G)$ is the size of a minimum covering set of edges in G.

Proposition: $\alpha(G) \leq \rho(G)$ in any graph G.

Theorem (Gallai 1): If G is a graph without loops, then $\alpha(G) + \tau(G) = |V(G)|$.

Theorem (Gallai 2): If G is a graph without isolated vertices, then $\nu(G) + \rho(G) = |V(G)|$.

Corollary (of König's theorem and Gallai's theorems): $\alpha(G) = \rho(G)$ in a bipartite graph G without isolated vertices.