Exercise-set 8.

1. In a school the students elect several committees. A student can be a member on several committees. Now every committee wants to select a president from its members. Every member of a committee is eligible for presidency, but the committees don’t want to share presidents (i.e., one person can be a president of at most one committee). When can this be attained?

2. a) In an Indian tribe there are 7 girls (A,B,...,G) and 6 boys (H,I,...,M) to be married. The chieftain made the table below about the possible couples. Can he find a wife for each of the boys?
   b) G and L don't want to get married anymore. Solve the problem in this case as well.

   |   | A | B | C | D | E | F | G |
---|---|---|---|---|---|---|---|---|
H  |   | * | * | * | * |   |   | * |
I  | * | * | * | * | * |   |   |   |
J  |   | * | * |   | * |   |   |   |
K  | * | * |   | * |   | * |   |   |
L  |   | * |   | * | * |   |   |   |
M  |   |   |   | * | * |   |   |   |

3. (a) Show that in an $r$-regular bipartite graph $|A| = |B|$.
   (b) Show that an $r$-regular bipartite graph satisfies Hall’s condition.
   (c) Show that an $r$-regular bipartite graph has a perfect matching.

4. There are $n$ couples on a hike. They want to distribute $2n$ different chocolate bars among themselves (so that everybody gets one). We know that everybody likes at least $n$ kinds from the $2n$ types, and each kind of chocolate is liked by at least one person in each couple. Prove that the chocolate bars can be distributed in such a way that everybody gets a type that he/she likes.

5. (MT'08) Suppose that the bipartite graph $G$ on $2n$ vertices has $n$ vertices in both of its classes, and that the degree of each vertex of $G$ is more that $\frac{n}{2}$. Show that $G$ contains a perfect matching.

6. (MT+'10) Each class of a bipartite graph contains exactly 5 vertices, and the degree of each vertex is at least 2. Show that this doesn’t imply that the graph contains a perfect matching.

7. Let $G$ be a simple, connected bipartite graph with $n$ vertices in both of its vertex classes, and let all the degrees in one class be different. Show that $G$ contains a perfect matching.

8. a) In a bipartite graph on 20 vertices 18 vertices have degree 5, and the degree of the other 2 vertices is 3. Show that the graph contains a perfect matching.
   b) In a bipartite graph on 19 vertices 17 vertices have degree 6, and the degree of the other 2 vertices is 3. Show that the graph contains a matching of 9 edges.

9. (MT'14) Let the two vertex classes of the bipartite graph $G(A, B; E)$ be $A = \{a_1, a_2, \ldots, a_8\}$ and $B = \{b_1, b_2, \ldots, b_8\}$. For each $1 \leq i, j \leq 8$ let $a_i$ and $b_j$ be adjacent if the entry in the $i$th row and $j$th column of the matrix below is 1. Determine whether $G$ contains a perfect matching or not.

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

10. Somebody selected 32 squares on a $(8 \times 8)$ chessboard in such a way that each row and each column contains exactly four selected squares. Show that we can select 8 out of the 32 squares in such a way that each row and each column contains exactly one of them.

11. Somebody divided a pack of 52 cards into 13 sets of 4 cards each at random. Prove that we can select one card from each set in such a way that we select exactly one of each of the 13 figures.

12. (**) (MT'19) In a simple bipartite graph on 20 vertices the degree of each vertex is either 5 or 6. Show that the graph contains a perfect matching.
13. (MT’15, MT++’19) Let the two vertex classes of the bipartite graph \( G(A, B; E) \) be \( A = \{a_1, a_2, \ldots, a_9\} \) and \( B = \{b_1, b_2, \ldots, b_9\} \). For each \( 1 \leq i \leq 9 \) and \( 1 \leq j \leq 9 \) let \( a_i \) and \( b_j \) be adjacent if the entry in the \( i \)th row and \( j \)th column of the matrix below is 1. Determine a maximum matching and a minimum covering set in \( G \).

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

14. (MT’16) Let the two vertex classes of the bipartite graph \( G(A, B; E) \) be \( A = \{a_1, a_2, \ldots, a_{101}\} \) and \( B = \{b_1, b_2, \ldots, b_{101}\} \). For each \( 1 \leq i \leq 101 \) and \( 1 \leq j \leq 101 \) let \( a_i \) and \( b_j \) be adjacent if \( i \cdot j \) is even. Determine \( \nu(G) \), the maximum number of independent edges, \( \rho(G) \), the minimum number of covering edges, and give a maximum matching and a minimum covering set of edges in \( G \).

15. Determine a maximum matching in each of the graphs below. Show that they are really maximum!

![Graphs](image)

16. (MT’18) Let the vertex set of the simple graph be \( V(G) = \{1, 2, \ldots, 10\} \). Let the vertices \( x, y \in V(G) \) be adjacent if and only if \( |x - y| = 3 \) or \( |x - y| = 5 \). Delete the edge \( \{3, 8\} \) from the graph \( G \), and denote the graph obtained by \( H \).

a) Determine \( \nu(H) \), the maximum number of independent edges in \( H \) and determine a maximum matching in \( H \).

b) Determine \( \alpha(H) \), the maximum number of independent vertices in \( H \) and determine a maximum independent set of vertices in \( H \).

17. (MT++’18) Let the two vertex classes of the bipartite graph \( G(A, B; E) \) be \( A = \{a_1, a_2, \ldots, a_8\} \) and \( B = \{b_1, b_2, \ldots, b_8\} \). For each \( 1 \leq i, j \leq 8 \) let \( a_i \) and \( b_j \) be adjacent if the entry in the \( i \)th row and \( j \)th column of the matrix below is 1. Determine \( \tau(G) \), the minimum number of covering vertices and \( \rho(G) \), the minimum number of covering edges, and give a minimum covering set of vertices and a minimum covering set of edges in \( G \).

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

18. a) The complement of a simple graph \( G \) on 100 vertices contains a perfect matching. Show that \( G \) can be colored with 50 colors.

b) (MT’10) In a simple graph \( G \) on 100 vertices the degree of each vertex is 55. Determine the chromatic number number of \( G \) if we know that the complement of it is a bipartite graph.