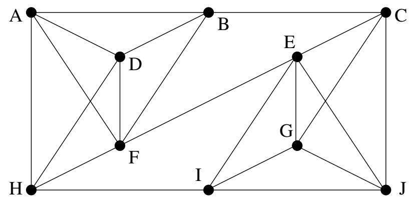
Introduction to the Theory of Computing 2.

Exercise-set 11.

1. At most how many pairwise edge-disjoint and vertex-disjoint paths are there between the following points in the graph below:
   a) B and I, b) A and J, c) B and H.



2. (MT'12) The graph G on 15 vertices is constructed from three cycles, on 4, 5 and 6 vertices each, in such a way that each vertex of the 5-vertex cycle was connected (with one edge) to all the other vertices of the other two cycles. Let s be a vertex of the 4-vertex cycle, and t be a vertex of the 6-vertex cycle.
   a) At most how many pairwise vertex-disjoint paths are there in G between s and t?
   b) At most how many pairwise edge-disjoint paths are there in G between s and t?

3. Determine the vertex- and edge connectivity numbers ($\kappa(G)$ and $\lambda(G)$) of the following graphs:
   a) the graph consisting of the vertices and edges of a cube,
   b) the complete bipartite graph $K_{m,n}$, where $m \geq n$,
   c) the graph in Exercise 1.

4. The vertices of an 18-vertex graph G can be divided into 3 classes of six vertices each, in such a way that 2 vertices are adjacent if and only if they are in different classes. Determine the largest integer k for which G is k-vertex-connected ($\kappa(G)$), and the largest integer l for which G is l-edge-connected ($\lambda(G)$).

5. Show that a k-(vertex-)connected graph G on n vertices has at least $kn/2$ edges.

6. Construct a simple graph which is 2-vertex-connected, 3-edge-connected and has minimum degree 4.

7. (MT'14) We connect two disjoint complete graphs on 5 vertices with 3 edges, in such a way that the resulting graph G is simple. Is it true in all cases that G is
   a) 3-(vertex)-connected;
   b) 3-edge-connected?

8. Show that if a graph is 3-(vertex-)connected, then it contains a cycle of even length.

9. (MT'07) Let G be a 3-(vertex-)connected graph with 100 vertices and let $x, y \in V(G)$ be two different vertices. Show that there is a path from x to y whose length (i.e. the number of edges in it) is not greater than 33.

10. (MT+'07) Prove that a graph G is 2(-vertex)-connected if and only if for every pair of vertices $x, y \in V(G)$ there is a cycle going through x and y.

11. a) Let G be a k-connected graph, and G' be a graph obtained by adding a new vertex of degree at least k to G. Show that if G' is a simple graph, then it is k-(vertex-)connected as well.
    b) Let G be a k-connected graph, and $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ be two disjoint point sets in it. Prove that there are k (completely) vertex-disjoint paths in G connecting A and B.

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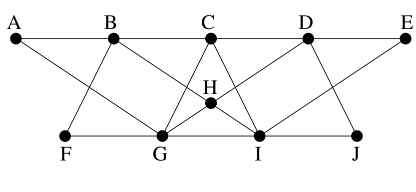
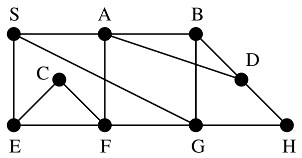
12. Can the vertices of the graph below be reached in the following order using the BFS algorithm?
    a) H, B, D, G, I, C, A, F, J, E b) F, B, A, G, C, H, I, D, E, J
    c) J, D, I, C, E, G, H, A, F, B d) A, B, G, C, H, F, I, D, E, J



13. (MT'15) The BFS algorithm visited the vertices of the graph below in the following order: $S, \square, \square, \square, H, \square, F, C, \square$.
a) Complete the sequence with the missing vertices (which are denoted by $\square$), and determine the corresponding BFS tree.
b) Can the edge $\{D, H\}$ be contained in an arbitrary BFS spanning tree started from S?



14. We want to decide for a given graph G and vertex s whether there is a cycle in G containing s and if yes then we want to find a shortest such cycle. Modify the BFS algorithm so that it can solve this problem as well.

15. In the connected graph G the degree of each vertex is 3. We start a BFS algorithm from vertex s which reaches vertex v in the 13th place (we consider s to be the vertex first reached). Is it possible that the distance of v from s is
a) 2, b) 3, c) 8?

16. (MT++'15) We call the spanning tree F of a connected graph G *suitable* for a vertex v of G, if there is a BFS started from v which is exactly F. At most how many edges can a connected graph G on 100 vertices have if it has a spanning tree which is suitable for every vertex of G?

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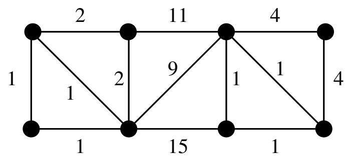
17. Determine a minimum weight spanning tree in the weighted graph below. How many such trees are there?



18. Let G be the complete graph on the vertex set $V(G) = \{1, 2, \ldots, 100\}$. For every $1 \leq i, j \leq 100$, $i \neq j$ let the weight of the edge $\{i, j, \}$ be the larger of the values of i and j. What is the weight of a minimum weight spanning tree in G? Determine such a tree. How many minimum weight spanning trees are there?

19. Let G be the complete graph on the vertex set $V(G) = \{1, 2, \ldots, 100\}$. For every $1 \leq i, j \leq 100$, $i \neq j$ let the weight of the edge $\{i, j, \}$ be 1, if $i, j \leq 50$, 2, if $i, j \geq 51$, and 3 for all the other edges. What is the weight of a minimum weight spanning tree in G? Determine such a tree.

20. (MT'15) Let G be a connected graph and $w : E(G) \to \mathbf{R}$ be a weight function on the edges of G. Suppose that one of the endpoints of the edge e of G is v and for all the edges f which are incident to v the inequality $w(e) \leq w(f)$ holds. Show that G has a minimum weight spanning tree which contains e.

21. (MT+'15) Let G be a connected graph and $w : E(G) \to \mathbf{R}$ be a weight function on the edges of G. Furthermore, let C be a cycle in G and e an edge of C. Suppose that $w(e) \geq w(f)$ holds for all the edges f of the cycle C. Show that G has a minimum weight spanning tree which doesn't contain e.