Introduction to the Theory of Computing 2.

Exercise-set 1.

1. In an ice cream parlor 26 kinds of ice cream are sold: Apple, Banana, Chocolate,.... In how many ways can a customer be served if he asks for the following (but leaves everything else to the attendant)?
   a) He wants 5 scoops in a cone, but he doesn't want 5 of the same flavor.
   b) He wants 1 scoop of A, 2 scoops of B, 1 scoop of C and 3 scoops of D in a cone.
   c) He wants 5 arbitrary scoops in a bowl, but no E among them.
   d) He wants 5 different scoops in a cone, and an F among them.
   e) He wants 5 different scoops in a bowl, and a G among them.
   f) He wants 5 arbitrary scoops in a cone, and at least one K among them.
   g) He wants 5 different scoops in a bowl, but if there is a P then there should be no R among them.
   h) He wants 35 scoops in a bowl, so that every flavor should appear among them, but none of them should appear more than twice.
   i) He wants an arbitrary number of all different scoops in a bowl (in this case 0 or 26 scoops are also possible).

2. a) In how many ways can we send 25 different postcards to 5 of our friends during the summer? (Possibly some of them don't get any, even one of them can get them all.)
   b) (MT++'12) In how many ways can Santa Claus distribute 20 identical chocolates to 5 children? (There is no rule for the distribution of chocolates, even one child can get all of them. We consider two cases different if there is a child who got a different number of chocolates.)

3. a) A combination lock can be opened by keying in 6 different numbers between 1 and 30. We know that in the code the numbers appear in increasing order. With how many tries can the lock be opened for sure (i.e. how many such codes can be made)?
   b) How many tries are needed if in the code the numbers are not necessarily different (but all the other conditions are the same)?
   c) (MT++'16) How many sequences of length 4 are there which contain 4 different numbers from $1, 2, \ldots, 100$ such that the largest number is the first one and the others are in increasing order? (E.g. 25,5,10,16 is such a sequence.)

4. There are 16 classes in a highschool, the number of students in each of them is 30. Each class sends a delegation of 4 students to the school's student committee. In how many ways can this be done?

5. Aunt Maggie plays the lottery with fervor, she plays with 20 lottery tickets each week. (In the Hungarian lottery one has to check 5 numbers between 1 and 90.) In how many ways can she fill in her tickets if
   a) she doesn't want identically filled out tickets,
   b) she fills the tickets out at random,
   c) she doesn't want identically filled out tickets, moreover 7 should appear on exactly 7 tickets,
   d) she only wants that if 13 appears on a ticket, then 7 should appear as well?

6. (MT+'16) In an ice cream parlor 26 kinds of ice cream are sold. A customer orders 4 bowls of ice cream, all for himself. He wants 3 scoops in each of the 4 bowls, and within a bowl all different kinds. He doesn't mind getting the same flavor more than once in different bowls, but he doesn't want two bowls with 3 identical scoops. In how many ways can the customer be served? (Within a bowl the order of the scoops doesn't matter, neither does the order of the bowls, since all of them are ordered by the same customer.)

7. (MT+'12) a) The local government of a small town has 20 members. They want to select committees, one for each of three different tasks (call them A, B and C). All the three committees have 4 members and a representative can serve on many committees but they want to avoid that all the four members of two committees are the same. In how many ways can the committees be selected?
   b) Another objective is that the mayor (who is one of the 20 members) should serve on at least one committee. In how many ways can the committees be selected in this case?
   c) And if the mayor wants to serve on exactly one committe (but it doesn't matter on which one)?

8. (MT+'10) Mr. X forgot his password and now wants to guess it. He remembers the following:
   (a) The password consists of 9 characters, each of whose is one of the 26 letters of the English alphabet (all of them are uppercase).
   (b) The password contains exactly 4 different letters.
   (c) The first 4 letters of the password are all different.
   (For example, Mr. X's password could be GRHXRRXGR.) How many passwords are there which satisfy the above conditions?

9. (MT++'10) Mr. Y forgot his password and now wants to guess it. He remembers the following:
(a) The password consists of 11 characters, each of whose is one of the letters A, B, C, D, E and F.
(b) One of the above six letters is repeated 3 times, three letters are repeated twice and the rest occurs only once in the password.
(For example, Mr. X's password could be DCABDFFEDBA.) How many passwords are there which satisfy the above conditions?

10. (MT'11) In how many ways can we choose 10 people out of the members of 15 married couples in such a way that we choose exactly 3 couples?

11. (MT'15) How many sequences of letters of length 12 can be made using the 26 letters of the English alphabet which contain exactly 4 X's and (exactly) 3 Y's?

12. (MT+'15) How many 5-element subsets does the set $\{1, 2, \ldots, 10\}$ have which contain more even numbers than odd numbers?

13. (MT'16) The neptun code of a student is a sequence consisting of 6 characters, each of which is either one of the 26 letters of the English alphabet or one of the digits $0, 1, \ldots, 9$. How many neptun codes are there which contain at most two digits?

14. (MT'17) In an imaginary country the license plate numbers consist of 6 characters, each of which is either one of the 26 letters of the English alphabet or one of the digits $0, 1, \ldots, 9$. Three characters must be letters and three must be numbers, with the condition that if three letters stand next to each other then all of them cannot be the same (e.g. a good license plate number is 37AAG1, but ABCD85 and 35HHH2 are not appropriate). How many license plate numbers can be given in the country?

15. (MT++'17) How many seven-digit integers are there in which the digit 8 occurs exactly three times?

16. * a) Does a set of size 99 have more even or odd subsets?
b) Does a set of size 100 have more even or odd subsets?

17. * At most how many subsets are there in a set of size 101 such that every two subsets have a common element?

18. Determine the value of the expressions below (for two decimal places).
a) (MT'10) $\log_2\left[\binom{101}{0}+\binom{101}{1}+\binom{101}{2}+\cdots+\binom{101}{50}\right]$
b) $\log_3\left[1\cdot\binom{100}{0}+2\cdot\binom{100}{1}+4\cdot\binom{100}{2}+\cdots+2^{100}\cdot\binom{100}{100}\right]$
c) (MT++'10) $\log_2\left[1\cdot\binom{32}{1}+2\cdot\binom{32}{2}+3\cdot\binom{32}{3}+\cdots+31\cdot\binom{32}{31}+32\cdot\binom{32}{32}\right]$

19. * Write the sums below in a closed form:
a) $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots\pm\binom{n}{n}$
b) $\binom{10}{0}\binom{90}{30}+\binom{10}{1}\binom{90}{29}+\cdots+\binom{10}{10}\binom{90}{20}$