Colorings of Graphs

Vertex Coloring

Definition: The *coloring* (of the vertices) of a graph G is an assignment of colors to the vertices of G in such a way that adjacent vertices get different colors.

Remarks: 1. Instead of colors we can use numbers.
2. If G contains a loop, it cannot be colored.
3. Multiple edges don't matter in the vertex-coloring.

Definition: The *chromatic number* of the graph G, $\chi(G)$ is the minimum number of colors needed to color the vertices of G.
In other words, $\chi(G) = k$ if and only if G can be colored using k colors, but it cannot be colored with $k-1$ colors.

Proposition: $\chi(G) = 1$ if and only if G is the empty graph (i.e. $E(G) = \emptyset$).

Theorem: $\chi(G) = 2$ if and only if G is bipartite (and not empty).

Definition: a graph $G = (V, E)$ is bipartite, if the vertices of it can be partitioned into two sets A, B (i.e. $V(G) = A \cup B$, s.t. A and B are disjoint), s.t. edges only go between A and B (and not inside A or B).

Theorem: A graph is bipartite if and only if it contains no odd cycles.

Lower bounds on the chromatic number

1: $\chi(G) \geq \omega(G)$, where $\omega(G)$ is the clique-number of G, i.e. the size of the largest cligue in G, where a clique is a complete subgraph of G.

This bound is sharp in many cases, e.g. complete graphs, bipartite graphs, interval graphs. (A graph is an interval graph if the vertices of it are closed intervals on the real line, and and two vertices are adjacent if and only if the corresponding intervals intersect.)

But for some graphs it can be very bad:
Theorem (Myczielski's constuction): There is an infinite sequence of graphs, $G_2, G_3, \ldots$ with the property that $\chi(G_k) = k$, but $\omega(G_k) = 2$ for all $k = 2, 3, \ldots$.

2: $\chi(G) \geq |V(G)|/\alpha(G)$, where $\alpha(G)$ is the independence number of G, i.e. the size of the largest independent set of vertices in G, where an independent set is an induced empty subgraph of G, i.e. a set of pairwise non-adjacent vertices in G.

Upper bounds on the chromatic number

1: $\chi(G) \leq |V(G)|$.

This is sharp only for the complete graph.

2: $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree in G. (this can be proved using the greedy coloring.)

This bound is sharp for the complete graph and odd cycles, but for no other (connected) graph.

An upper bound for planar graphs

Theorem (4-color theorem, Appel and Haken): Every planar graph can be colored using (at most) 4 colors.

Edge Coloring

Definition: The *edge-coloring* of a graph G is an assignment of colors to the edges of G in such a way that adjacent edges get different colors.

Remarks: 1. Instead of colors we can use numbers.
2. If G contains a loop, it cannot be colored.
3. Multiple edges do matter in the edge-coloring! (They all have to get different colors.)

Definition: The *edge-chromatic number* of the graph G, $\chi_e(G)$ is the minimum number of colors needed to color the edges of G.
In other words, $\chi_e(G) = k$ if and only if the edges of G can be colored using k colors, but they cannot be colored with $k-1$ colors.

Lower bound on the edge-chromatic number

Proposition: $\chi_e(G) \geq \Delta(G)$.

Upper bounds on the edge-chromatic number

Theorem (Vizing): If G is a simple graph, then $\chi_e(G) \leq \Delta(G) + 1$.

Theorem (Shannon): If G is an arbitrary graph, then $\chi_e(G) \leq 3/2 \cdot \Delta(G)$.

Theorem (König): If G is a bipartite graph, then $\chi_e(G) = \Delta(G)$.