

# Introduction to the Theory of Computing 1.

## Second Retake of the Second Midterm Test

December 18, 2023

1. We know that  $x_1 = x_2 = x_3 = x_4 = 1$  is a solution of the system of equations below, but it has no such solutions in which the value of one of the variables is 2. Determine all the solutions of the system of equations.

$$\begin{aligned}x_1 + 2x_2 + 2x_3 - 2x_4 &= 3 \\2x_1 + 3x_2 - 2x_3 + x_4 &= 4 \\x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\3x_1 + 3x_2 + x_3 + x_4 &= 8\end{aligned}$$

2. Let  $A$  be the matrix below. Is there such a real number  $p$  for which  $\det A = 0$  independently of the value of  $q$ ?

$$A = \begin{pmatrix} q & 2 & 1 & 1 \\ 2 & p & 3 & 0 \\ 3 & 5 & 2 & 0 \\ 3 & 1 & 2 & 1 \end{pmatrix}$$

3. All we know of the  $5 \times 5$  matrix  $A$  and of the vectors  $\underline{b}, \underline{c} \in \mathbf{R}^5$  that the matrix equations  $A\underline{x} = \underline{b}$  and  $A\underline{x} = \underline{c}$  have a different number of solutions. How many solutions does the matrix equation  $A\underline{x} = \underline{0}$  have?
4. Let  $A$  be the matrix below. Does there exist a matrix  $B$  for which  $A \cdot B = I$  holds, where  $I$  is the  $2 \times 2$  identity matrix? If yes, then determine one such matrix.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 0 & 6 \end{pmatrix}$$

5. Determine the rank of the matrix below depending on the parameter  $p$ .

$$\begin{pmatrix} 1 & 3 & p & p & 2 \\ 2 & 1 & 3 & p & 3 \\ 1 & 4 & 1 & p & 5 \end{pmatrix}$$

6. \* Let  $A$  be an  $n \times n$  matrix of rank  $n$ , and let  $k$  and  $m$  be positive integers for which  $k + m \geq n$ ,  $k \leq n$  and  $m \leq n$  holds. Prove that  $A$  is the sum of a matrix of rank  $k$  and a matrix of rank  $m$  (that is, there exist matrices  $K, M$  of ranks  $k$  and  $m$  respectively, for which  $K + M = A$  holds).

Please work on stapled sheets only, and submit all of them at the end of the midterm, including drafts. Write your name on every sheet you work on, and write your Neptun code on the first page.

You have 90 minutes to work on the problems. Each of them is worth 10 points. The problem marked with an \* is supposed to be more difficult.

The details of the solutions must be explained, giving the result only is not worth any points. Notes, calculators (or similar devices) cannot be used.

Please turn over for the first midterm.