Introduction to the Theory of Computing 1.
Second Midterm Test

December 1, 2023

1. Determine the number of solutions for every value of the parameters p and q of the system of equations below (you don't need to determine the solutions themselves).

$$\begin{aligned} -x_1 - 4x_2 + 3x_3 + 5x_4 &= 0 \\ 2x_1 + 5x_2 - 24x_3 + (3p-1)x_4 &= 0 \\ 3x_1 + 17x_2 + 22x_3 + (q-5p)x_4 &= 0 \end{aligned}$$

2. Let $\pi = (5, 2, 6, 1, 4, 3)$ (that is, π is that permutation of the numbers $1, 2, \ldots, 6$ for which $\pi_1 = 5, \pi_2 = 2, \ldots, \pi_6 = 3$). Let the entry in the ith row and jth column of the 6×6 matrix A be

$$a_{ij} = \begin{cases} 1, & \text{if } j \geq \pi_i, \\ 0, & \text{if } j < \pi_i \end{cases}$$

for each $1 \leq i, j, \leq 6$ (that is, for each i, in the ith row on the π_ith place and to the right of it the entry is 1, to the left of it it is 0). Evaluate the determinant of A *using the original definition.* (So don't use any properties of the determinant, or theorems about it during the solution, but determine the value using the definition only.)

3. Evaluate the determinant below.

$$\begin{vmatrix} 4 & -3 & 1 & 1 & 1 \\ -8 & 6 & 17 & 19 & 23 \\ 12 & -9 & 1 & 1 & 1 \\ -5 & 4 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 \end{vmatrix}$$

4. a) For each value of the parameter p decide whether the matrix A below is invertible or not, and if yes, then determine A^{-1}.
b) For those p's for which A^{-1} exists determine the matrix $(A^{-1} - I)(A^2 - A) + (A - I)^2$.

$$A = \begin{pmatrix} 2 & 1 \\ 2p & p+1 \end{pmatrix}$$

5. At most how many vectors can be chosen from the vectors $\underline{x}, \underline{y}, \underline{z}, \underline{u}, \underline{v}$ in $\mathbf{R}^4$ below in such a way that the chosen vectors are linearly independent?

$$\underline{x} = (-2, 3, 2, -1,)^T, \ \underline{y} = (-14, 21, 15, -2)^T, \ \underline{z} = (2, -3, -1, 6)^T,$$
$$\underline{u} = (4, -6, -1, 17)^T, \ \underline{v} = (10, -11, -11, -3)^T$$

6. * Suppose that $A + A^2 + A^3 = 0$ holds for the $n \times n$ matrix A (here 0 is the zero matrix). Show that in this case either $\det(A) = 0$ or $\det(A) = 1$.

Please work on stapled sheets only, and submit all of them at the end of the midterm, including drafts.

Write your name on every sheet you work on, and write your Neptun code and the number of the group you are registered to in Neptun (A1, A2 or A3) on the first page.

You have 90 minutes to work on the problems. Each of them is worth 10 points. To obtain a signature you have to achieve at least 24 points on each of the two midterm tests.

The details of the solutions must be explained; giving the result only is not worth any points. Notes, calculators or any additional tools cannot be used. The problem marked with an * is supposed to be more difficult.