

**Introduction to the Theory of Computing 1.**  
**Second Repeat of the Second Midterm Test**  
2018. December 17.

1. Evaluate the determinant below *using the original definition*. (So don't use any properties of the determinant, or theorems about it during the solution, but determine the value using the definition only.)

$$\begin{vmatrix} 0 & 7 & 2 & 0 & 0 \\ 4 & 8 & 9 & 0 & 1 \\ 2 & 5 & 1 & 3 & 7 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 9 & 3 & 0 & 5 \end{vmatrix}$$

2. Solve the matrix equation  $X \cdot A = B$  for the matrices  $A$  and  $B$  below, i.e. determine all the matrices  $X$  for which  $X \cdot A = B$  holds.

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 0 & 2 \\ 12 & 14 & 2 & 12 \\ 3 & 3 & 5 & 8 \end{pmatrix} \qquad B = ( 15 \quad 18 \quad 21 \quad 35 )$$

3. For the matrix  $A$  with 2018 columns it holds that there are two vectors,  $\underline{x}_1, \underline{x}_2 \in \mathbf{R}^{2018}$ ,  $\underline{x}_1 \neq \underline{x}_2$  for which  $A \cdot \underline{x}_1 = A \cdot \underline{x}_2$  holds. Show that in this case there exist vectors  $\underline{z}_1, \underline{z}_2, \dots, \underline{z}_{2018} \in \mathbf{R}^{2018}$ , such that no two of them are equal, and  $A \cdot \underline{z}_1 = A \cdot \underline{z}_2 = \dots = A \cdot \underline{z}_{2018}$ .
4. For each value of the parameters  $p$ ,  $q$  and  $r$  decide whether the matrix below is invertible or not, and if yes, determine the upper left entry of the inverse matrix.

$$\begin{pmatrix} p & 0 & 0 \\ q & 3 & 8 \\ r & 4 & 11 \end{pmatrix}$$

5. Determine the rank of the matrix  $A$  in the second question.
6. \* Let  $M$  be a matrix with 100 columns. Let  $A$  be the matrix consisting of the first 70 columns of  $M$ ,  $B$  the matrix consisting of the last 70 columns of  $M$ , and  $X$  the matrix consisting of the middle 40 columns of  $M$ . Prove that  $r(A) + r(B) \geq r(M) + r(X)$ , where  $r$  denotes the rank of the matrices.

The full solution of each problem is worth 10 points. Show all your work! Results without proper justification or work shown deserve no credit. Calculators (or other devices) are not allowed to use. The question denoted by an \* is supposed to be more difficult.