Exercise-set 6.
Solutions

1. a) No (lin. dependent).
   b) Yes, $[\underline{a}]_B = (1, -1, 2, 0)^T$.
2. a) Basis e.g.: $\underline{u}_1 = (3, 1, -1)^T$, $\underline{u}_2 = (6, -1, -4)^T$ (two vectors in the plane), $\dim V = 2$.
   b) Basis e.g.: $\underline{u}_1 = (1, 0, 0, -4)^T$, $\underline{u}_2 = (0, 1, 0, -3)^T$, $\underline{u}_3 = (0, 0, 1, -2)^T$, $\dim V = 3$.
3. a) E.g.: $\underline{u}_1 = \underline{v}$, $\underline{u}_2 = (0, 1, 0, 1)^T$, $\underline{u}_3 = (0, 0, 1, -1)^T$, $\dim V = 3$.
   b) $\underline{u}_1 = \underline{v}$, $\dim V = 1$.
4. Fourth vector e.g.: $\underline{z} = (0, 0, 0, 1)^T$, $[\underline{a}]_B = (4, -3, 1, 0)^T$.
5. E.g.: $\underline{u}_1 == (1, 1, 0, 0)^T$, $\underline{u}_2 = (0, 0, 3, 1)^T$, ($\dim V = 2$).
6. $\dim V = 2$.
7. $\dim V = 2$, a basis e.g.: $\underline{u}_1 = (1, 0, -1, -3)^T$, $\underline{u}_2 = (0, 1, 1, -2)^T$.
8. V is a subspace, $\dim V = 2$, a basis e.g. $\underline{u}_1 = (1, 1, 1, 1, 1)^T$, $\underline{u}_2 = (1, 2, 4, 8, 16)^T$.
9. E.g.: $\underline{v}_2 == (0, 1, 0, 2)^T$, $\underline{v}_3 = (0, 0, 1, -3)^T$, ($\dim V = 3$).
10. $p = -1$, $[\underline{v}]_B = (7, -4)^T$.
11. Linearly dependent, $\dim V = 3$.
12. 99 (the first 99 vectors are linearly independent, and the last one is a linear combination of these).
13. No such value (4 vectors in a subspace of dimension 3 are always linearly dependent).
14. $p \neq 1$.
15. No. The vectors are linearly dependent, $\dim V$ can be 3 or 4.
16. (MT++'17) a) Yes, they are linearly independent.
    b) Yes, they are linearly independent.
17. Yes, they must be linearly independent.
18. Any 4 non-parallel vectors in a 2-dimensional subspace, e.g. $\underline{u}_1 = (1, 0, 0, 0)^T$, $\underline{u}_2 = (0, 1, 0, 0)^T$, $\underline{u}_3 = (1, 1, 0, 0)^T$, $\underline{u}_4 = (-1, 1, 0, 0)^T$.
19. 5 linearly dependent vectors in a 3-dimensional subspace, e.g. $\underline{u}_1 = (1, 0, 0, 0, 1)^T$, $\underline{u}_2 = (0, 1, 0, 0, 0)^T$, $\underline{u}_3 = (0, 0, 1, 0, 0)^T$, $\underline{u}_4 = (1, 1, 1, 0, 0)^T$, $\underline{u}_5 = (1, 2, 3, 0, 0)^T$.
20. No, we can always select a basis (of 4 vectors) from a generating system.
21. By contradiction: otherwise two bases from the subspaces would give 100 linearly independent vectors in $\mathbf{R}^{99}$, a contradiction.