Introduction to the Theory of Computing 1.

Exercise-set 9.

1. Evaluate the following determinant *using the expansion theorem.*

$$\begin{vmatrix} 4 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 5 & \pi \\ 3 & 1 & 1 & 1 \end{vmatrix}$$

2. (-, MT'15, MT+'17, MT++'17) Evaluate the following determinants for all values of the parameter p.

a) $\begin{vmatrix} 1 & p & 2 & p \\ 1 & 0 & 1 & p \\ 1 & 0 & 2 & p \\ 1 & 1 & p & 4 \end{vmatrix}$ b) $\begin{vmatrix} p & 2 & 3 & p \\ 5 & p & 0 & 0 \\ p & 0 & 4 & p \\ 8 & 5 & p & 8 \end{vmatrix}$ c) $\begin{vmatrix} p & 1 & 3 & 7 \\ 1 & p & 8 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & p & p \end{vmatrix}$ d) $\begin{vmatrix} p & 2p & p & 3p \\ 3 & 9 & 3 & 6 \\ 1 & 2 & 7 & 1 \\ 3 & 7 & 8 & 4 \end{vmatrix}$

3. Evaluate the determinants of the following matrices:

a) MT+'09 $\begin{pmatrix} 2 & 0 & 0 & 1 \\ 3 & 2 & 0 & 0 \\ 4 & 3 & 2 & 0 \\ 5 & 4 & 3 & 2 \end{pmatrix}$ b) MT++'09 $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 0 \\ 3 & 2 & 0 & 0 \\ 7 & 8 & 9 & 8 \end{pmatrix}$

4. Evaluate the determinants below (by any method).

a) MT'14 $\begin{vmatrix} 1 & -1 & 0 & 3 & 0 \\ -1 & 1 & 3 & 6 & 5 \\ -3 & 3 & 2 & -3 & 2 \\ 3 & -1 & 4 & 9 & -8 \\ -4 & 4 & 1 & -8 & 3 \end{vmatrix}$ b) MT++'15 $\begin{vmatrix} 1 & -3 & 2 & 3 & 0 \\ 2 & -6 & 4 & 11 & -7 \\ -3 & 9 & -5 & -9 & 5 \\ 4 & -12 & 8 & 13 & -2 \\ 5 & -11 & 10 & -1 & -12 \end{vmatrix}$

5. In a $2k \times 2k$ matrix all the entries on the main diagonal are c, and all the entries on the side diagonal are d. Evaluate the determinant of the matrix.

6. We multiply each entry of an $n \times n$ matrix A by the cofactor belonging to it. What is the sum of the n^2 terms obtained this way?

7. All the entries of an $n \times n$ matrix are fixed with the exception of one entry. Is it true that this entry can always be chosen in such a way that the matrix obtained has 0 determinant?

8. We call an entry of an $n \times n$ matrix with nonzero determinant *interesting*, if by changing this entry (and only this) the determinant of the matrix can be made 0.
a) Is it true that each entry of every matrix with nonzero determinant is interesting?
b) Is it true that there is an interesting entry in each row of a matrix with nonzero determinant?

9. MT+'07 The $n \times n$ matrix A satisfies the following property: for every entry $a_{i,i}$ in the main diagonal either all entries under $a_{i,i}$ are 0 or all entries to the right of $a_{i,i}$ are 0. Prove that $\det A = a_{1,1}a_{2,2} \cdot \ldots \cdot a_{n,n}$.

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10. Determine the equation of the plane through the points $A(1, 2, 12)$, $B(3, 1, 3)$ and $C(2, -1, -5)$.

11. MT++'12 Determine the equation of the plane through the point $P(2, -3, 4)$ containing the line given by the system of equations $\frac{x-1}{2} = \frac{y+4}{-7}$, $z = -5$.

12. MT'12 Determine the equation of the plane which passes through the points $P(1, 3, 4)$ and $Q(3, 6, 10)$ and is parallel to the line given by the system of equations $\frac{x-9}{3} = y + 4 = \frac{z}{5}$.

13. MT+'14 Determine whether the lines e and f given by the systems of equations below are parallel or not. If yes, then determine the equation of the plane S containing them.

$$e: \frac{2x-3}{4} = \frac{3y+4}{6} = \frac{z}{2} \qquad f: \frac{x+1}{2} = \frac{y-4}{2} = \frac{3z-5}{6}$$

14. Determine whether the points $A(2; -1; 1)$, $B(4; -2; -1)$, $C(-6; -11; 2)$ and $D(10; 15; 3)$ are on one plane in 3-space or not.

15. MT'14 For which values of the parameter p will the points $A(2, 3, 3)$, $B(3, 4, 1)$, $C(4, 6, 2)$ and $D(p, 2, 5)$ lie on one plane?

16. Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$. Determine whether the following matrix operations can be performed or not and if yes, calculate the result of them.
a) $2A + 3B$, b) $A \cdot B$, c) $B \cdot A$, d) $A \cdot B + 2B$, e) $B \cdot B^T$.

17. MT+'11 Let the entries in the ith row and jth column of the 4×4 matrices A and B be denoted by $a_{i,j}$ and $b_{i,j}$ for all $1 \leq i, j \leq 4$, respectively. Suppose that for all $1 \leq i, j \leq 4$

$$a_{i,j} = \begin{cases} i + j, & \text{if } j = 1, 2, \\ 9 - i - j, & \text{if } j = 3, 4; \end{cases} \qquad b_{i,j} = \begin{cases} j, & \text{if } i = 1, 3, \\ 1 - j, & \text{if } i = 2, 4. \end{cases}$$

(a) Determine the matrix $A \cdot B$.
(b) Evaluate the determinant of the matrix $B \cdot A$.

18. Determine whether the following hold for all $n \times n$ matrices A and B. (I denotes the $n \times n$ identity matrix.)
a) $AB + B = (A + I)B$, b) $(A + B)(A - B) = A^2 - B^2$, c) $(A + I)^2 = A^2 + 2A + I$.

19. Compute the following matrices. (A^n is the product with n factors, each of whose is A.)
a) $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}^{2014}$ b) $\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}^{2014}$ c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{2014}$

20. MT'08, MT+'08 a) Compute the 2008th power of the matrices A and B below.
b) What can we deduce about the determinants of the matrices from these results?

$$A = \begin{pmatrix} -2 & 1 & -2 \\ 1 & -2 & 2 \\ 2 & -2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 2 & -1 \\ 3 & 7 & -3 \\ 8 & 16 & -7 \end{pmatrix}$$

21. MT'15 a) Compute the matrix A^{2015} for the matrix A below.
b) Evaluate the determinant of B^{2015} for the matrix B below.

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}$$

22. MT'12 Determine which of the statements is true for any square matrix A. (Here I denotes the identity matrix and A^k means a product with k factors, each of whose is A.)
a) If there exist an integer $k \geq 1$ for which $A^k = I$ then $\det(A) = 1$ or $\det(A) = -1$.
b) If $\det(A) = 1$ or $\det(A) = -1$ then there exist an integer $k \geq 1$ for which $A^k = I$.
(If a statement is true, prove it, if not, give a counterexample.)

23. Determine all the 2×2 matrices X all of whose entries are rational numbers and for which $X^{2018} = \begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$ holds.

24. MT+'15 For the 5×3 matrix A it holds that the entry of the matrix $A \cdot A^T$ in the lower left corner is 2015. Determine the entry of $A \cdot A^T$ in the upper right corner.

25. Let A be a real symmetric matrix for which the diagonal entries of A^2 are all 0. Show that in this case A is the 0 matrix.

26. MT++'12 All the entries of the matrix A are 0, 1 or -1, and it has exactly 2012 nonzero entries. Determine the sum of the entries in the main diagonal of the matrix $A \cdot A^T$.

27. MT'14 For the $n \times n$ matrix A it holds that if we subtract 1 from each entry of its main diagonal (but we don't change the other entries) then we get a matrix with 0 determinant. Show that in this case if we subtract 1 from each entry of the main diagonal of A^2 then we get a matrix with 0 determinant as well.

28. MT'16 Are there two 2×2 matrices, A and B for which $A \cdot A = B \cdot B$, but $A \neq B$ and $A \neq -B$? (If the answer is no, prove it; if yes, give an example.)