Introduction to the Theory of Computing 1.

Exercise-set 8.

1. Determine the sign of the elementary product of the entries of the side diagonal of an $n \times n$ matrix as a function of n. (The side diagonal of the matrix is the line connecting the upper right corner of the matrix with the lower left corner.)

2. We placed n rooks on an $n \times n$ chessboard in such a way that they don't attack each other. To each such placement corresponds an $n \times n$ matrix in which there is a 1 at the position of the rooks and 0 at the other places. What is the value of the determinants of the matrices obtained this way?

3. Evaluate the determinants below *using the original definition.* (So don't use any properties of the determinant, or theorems about it during the solution, but determine the value using the definition only.)

a) (MT'17) $\begin{vmatrix} 0 & 0 & 1 & 2 & 5 \\ 3 & 0 & 6 & 8 & 9 \\ 0 & 0 & 5 & 0 & 0 \\ 5 & 4 & 7 & 3 & 2 \\ 0 & 0 & 2 & 0 & 1 \end{vmatrix}$ b) (MT+'15) $\begin{vmatrix} 9 & 8 & 5 & 7 & 4 \\ 0 & 2 & 0 & 0 & 6 \\ 0 & 3 & 0 & 2 & 7 \\ 0 & 1 & 0 & 0 & 3 \\ 1 & 9 & 0 & 8 & 6 \end{vmatrix}$

c) $\begin{vmatrix} 5 & 3 & 9 & 1 & 7 \\ 4 & 1 & 8 & 6 & 5 \\ 2 & 0 & 0 & 0 & 9 \\ 9 & 0 & 0 & 0 & 2 \\ 8 & 0 & 0 & 0 & 3 \end{vmatrix}$ d) MT'12 $\begin{vmatrix} 0 & 0 & 3 & 0 & 8 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{5} & 0 \\ 2 & 0 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 & 0 \end{vmatrix}$

4. (MT++'10) Suppose we evaluate the determinant below *using the original definition.*
a) How many nonzero terms will arise?
b) What will be the sign of the elementary product one of whose factors is 23?

$$\begin{vmatrix} 0 & 11 & 12 & 13 & 14 \\ 0 & 15 & 0 & 0 & 21 \\ 0 & 22 & 23 & 0 & 0 \\ 0 & 0 & 24 & 0 & 25 \\ 31 & 32 & 33 & 34 & 35 \end{vmatrix}$$

5. Show that the values of the following determinants are not zero without actually finding the exact values:

a) $\begin{vmatrix} 111 & 100 & 225 & 235 \\ 220 & 312 & 220 & 410 \\ 215 & 180 & 268 & 305 \\ 315 & 145 & 205 & 122 \end{vmatrix}$ b) $\begin{vmatrix} 1849 & 1444 & 1896 & 1222 \\ 1490 & 1703 & 1790 & 1526 \\ 1342 & 1566 & 1541 & 1514 \\ 1242 & 1552 & 1382 & 1825 \end{vmatrix}$

6. (MT'07) Let A be such a 40×40 matrix, whose upper left 18×23 submatrix contains all 0's. Show that $\det A = 0$.

7. (MT'04) In an 101×101 matrix A the entry in the ith row and jth column is

$$a_{ij} = \begin{cases} \text{the } (2i+j)\text{th digit of } 3^{2004}, & \text{if } i \cdot j \text{ is even,} \\ 0, & \text{if } i \cdot j \text{ is odd.} \end{cases}$$

Evaluate the determinant of A.

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8. Evaluate the determinants below.

a) (MT+'12) $\begin{vmatrix} 2 & 4 & 2 & 8 \\ 5 & 10 & 9 & 15 \\ 4 & 4 & 4 & 10 \\ 3 & 7 & 7 & 5 \end{vmatrix}$ b) $\begin{vmatrix} 2 & -2 & 4 & 6 \\ 1 & 1 & 0 & 7 \\ 2 & 0 & 4 & 8 \\ 4 & 1 & 6 & 20 \end{vmatrix}$ c) (MT'14) $\begin{vmatrix} 1 & -1 & 0 & 3 & 0 \\ -1 & 1 & 3 & 6 & 5 \\ -3 & 3 & 2 & -3 & 2 \\ 3 & -1 & 4 & 9 & -8 \\ -4 & 4 & 1 & -8 & 3 \end{vmatrix}$

d) $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 4 & 7 & 10 & 13 & 16 \\ 1 & 5 & 9 & 13 & 17 & 21 \\ 1 & 6 & 11 & 16 & 21 & 26 \\ 1 & 7 & 13 & 19 & 25 & 31 \end{vmatrix}$

9. (MT+'08) Evaluate the determinant of the matrix A below:

$$A = \begin{pmatrix} 0 & 3 & 4 & 5 \\ 2 & 0 & 4 & 5 \\ 2 & 3 & 0 & 5 \\ 2 & 3 & 4 & 0 \end{pmatrix}$$

10. In an $n \times n$ matrix A the entry in the ith row and jth column is a_{ij}. Evaluate the determinant of A if

a) $a_{ij} = \begin{cases} i, & \text{if } i = j, \\ 1, & \text{if } i \neq j \end{cases}$ b) $a_{ij} = i^2 j^2 + 1$ c) $a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{if } i \neq j \end{cases}$

d) $a_{ij} = i \cdot j$ e) $a_{ij} = \min\{i, j\}$ f) $a_{ij} = 2^i + 5j + 3$

11. Each entry of the $n \times n$ matrix ($n \geq 2$) is +1 or -1. Show that the determinant of the matrix is divisible by 2^{n-1}.

12. (MT'11) Determine the value of (det A + det B), where the matrices A and B are as below.

$$A = \begin{pmatrix} 1 & 2 & 5 & 3 \\ 1 & 4 & 3 & 7 \\ 3 & 9 & 13 & 19 \\ 1 & 4 & 6 & 23 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 7 & 8 & 16 \\ 1 & 4 & 3 & 7 \\ 3 & 9 & 13 & 19 \\ 1 & 4 & 6 & 23 \end{pmatrix}$$

13. (MT++'14) Show that the determinants of the two matrices below are equal without actually computing them:

$$A = \begin{pmatrix} 9 & 2 & 8 & 1 & 2 \\ 8 & 7 & 4 & 7 & 12 \\ 6 & 4 & 3 & 1 & 8 \\ 8 & 3 & 0 & 4 & 12 \\ 2 & 1 & 7 & 3 & 9 \end{pmatrix}, \qquad B = \begin{pmatrix} 9 & 8 & 8 & 4 & 2 \\ 2 & 7 & 1 & 7 & 3 \\ 6 & 16 & 3 & 4 & 8 \\ 2 & 3 & 0 & 4 & 3 \\ 2 & 4 & 7 & 12 & 9 \end{pmatrix}$$

14. In the determinant below a, b, c and d denote real numbers. Evaluate the determinant.

$$\begin{vmatrix} 1 & a & b & c+d \\ 1 & b & c & a+d \\ 1 & c & d & a+b \\ 1 & d & a & b+c \end{vmatrix}$$

15. How does the determinant of a 10×10 matrix change, if we perform the operations below?
a) We multiply each entry of the matrix by 2.
b) We add the difference of the third and seventh rows to these two rows (i.e. the third and seventh rows).
c) For each $1 \leq i, j \leq 10$ we multiply the entry in the ith row and jth column by $\frac{i}{j}$.

16. (MT'08) In each row of an 4×4-es matrix the sum of entries is 2008. Show that if we replace each entry in one column of this matrix by 1's, then the determinant of the matrix obtained is 1/2008 times the original determinant.

17. (MT+'17, MT++'17) Evaluate the determinants below for all real values p.

a) $\begin{vmatrix} p & 1 & 3 & 7 \\ 1 & p & 8 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & p & p \end{vmatrix}$ b) $\begin{vmatrix} p & 2p & p & 3p \\ 3 & 9 & 3 & 6 \\ 1 & 2 & 7 & 1 \\ 3 & 7 & 8 & 4 \end{vmatrix}$