Introduction to the Theory of Computing 1.

Exercise-set 7.

1. Solve the following systems of linear equations using Gaussian elimination (in part b) for all the values of the real parameter p).

   a)
   $$\begin{aligned}
   2x_1 + 10x_2 - 2x_3 + 4x_4 &= 2 \\
   5x_1 + 23x_2 - 9x_3 + 12x_4 &= -1 \\
   -x_1 + 11x_3 - 2x_4 &= 34 \\
   3x_1 + 17x_2 + x_3 + 7x_4 &= 21
   \end{aligned}$$

   b)
   $$\begin{aligned}
   2x_1 + 10x_2 - 2x_3 + 4x_4 &= 2 \\
   5x_1 + 23x_2 - 9x_3 + 12x_4 &= -1 \\
   -x_1 + 11x_3 - 2x_4 &= 34 \\
   3x_1 + 17x_2 + x_3 + 7x_4 &= p
   \end{aligned}$$

2. (MT+'05) Solve the following system of linear equations among the real numbers.

$$\begin{aligned}
x_1 + 2x_2 + 3x_3 + 4x_4 &= 14 \\
2x_1 + 6x_2 + 10x_3 + 6x_4 + 2x_5 &= 28 \\
x_1 + 5x_2 + 9x_3 + 2x_4 + 3x_5 &= 16
\end{aligned}$$

3. (MT'11) Determine for which values of the parameter c the system of equations below is consistent. If it has solutions, then determine all of them.

$$\begin{aligned}
x_1 + 2x_2 - 5x_3 + x_5 &= 4 \\
3x_1 + 8x_2 - 9x_3 + 2x_4 + 3x_5 &= 2 \\
-2x_1 - 5x_2 + 7x_3 + 2x_4 - 8x_5 &= 6 \\
2x_1 + 6x_2 - 4x_3 + c \cdot x_4 + (c-15) \cdot x_5 &= 13
\end{aligned}$$

4. (MT+'15) Determine for which values of the parameters p and q the system of equations below is consistent. If it has solutions, then determine all of them.

$$\begin{aligned}
x_1 - 3x_2 - 14x_3 &= -17 \\
2x_1 - 6x_2 - 28x_3 + p \cdot x_4 &= q - 34 \\
3x_1 - 7x_2 - 36x_3 + 4p \cdot x_4 &= 4q - 37
\end{aligned}$$

5. (MT'14, MT++'04, MT++'14) Determine for which values of the parameter p the system of equations below is consistent. If it has solutions, then determine all of them.

   a)
   $$\begin{aligned}
   x_1 - x_2 + 4x_4 &= -2 \\
   2x_1 - 2x_2 + x_3 + 8x_4 &= -3 \\
   x_1 + x_2 + 6x_3 + 8x_4 &= 2 \\
   3x_1 - 3x_2 + p \cdot x_3 + (p^2 + p + 12) \cdot x_4 &= -6
   \end{aligned}$$

   b)
   $$\begin{aligned}
   -x + 3y - z - 3w &= -2 \\
   2x - 6y + 5z + 12w &= 7 \\
   3x - 9y + 5z + pw &= 9
   \end{aligned}$$

   c)
   $$\begin{aligned}
   x_1 + 5x_2 - x_3 + 4x_4 &= 2 \\
   x_1 + 8x_2 + 8x_3 - 2x_4 &= 14 \\
   3x_1 + 13x_2 - 9x_3 + p \cdot x_4 &= -2 \\
   2x_1 + 14x_2 + 10x_3 + (p-13) \cdot x_4 &= 23
   \end{aligned}$$

6. (MT+'01, MT++'04) Determine (all) the points of intersection of the planes S_1, S_2 and S_3 below.

   a)
   $$\begin{aligned}
   S_1 &: & x + y + z &= 6 \\
   S_2 &: & 2x + 3y - 2z &= 0 \\
   S_1 &: & 5x + 7y - 3z &= 6
   \end{aligned}$$

   b)
   $$\begin{aligned}
   S_1 &: & 2x - y + 5z &= 3 \\
   S_2 &: & 3x + 2y + 6z &= 4 \\
   S_1 &: & 4x - 9y + 13z &= 9
   \end{aligned}$$

7. Which set of vectors forms a basis in $\mathbf{R}^4$? If it does, then determine the coordinate vector of $\underline{v} = (3, 5, 8, 9)^T$ in the given basis.
   a) $(2, 5, -1, 3)^T$, $(10, 23, 0, 17)^T$, $(-2, -9, 11, 1)^T$, $(4, 12, -2, 7)^T$,
   b) $(1, 3, 2, 1)^T$, $(1, 4, 3, 2)^T$, $(1, 4, 5, 4)^T$, $(1, 4, 5, 5)^T$.

8. (MT+'11) Determine for all values of the parameter p whether the statement $\underline{d} \in \langle \underline{a}, \underline{b}, \underline{c} \rangle$ holds for the vectors $\underline{a}, \underline{b}, \underline{c}, \underline{d} \in \mathbf{R}^4$:
   $\underline{a} = 3, -1, 2, 1)^T$, $\underline{b} = (15, 8, 8, 7)^T$, $\underline{c} = (12, 6, 7, p)^T$, $\underline{d} = (6, 8, -9, 12)^T$.

9. (MT+'08) For which values of the parameter p will the set of vectors $B = \{\underline{b}_1, \underline{b}_2, \underline{b}_3, \underline{b}_4\}$ form a basis in $\mathbf{R}^4$? For these values of p determine the coordinate vector $[v]_B$.
   $\underline{b}_1 = (\sqrt{2}, 2, 11, 13)^T$, $\underline{b}_2 = (\sqrt{2}, 9, 15, 12)^T$, $\underline{b}_3 = (\sqrt{2}, -19, 4, 19)^T$, $\underline{b}_4 = (\sqrt{2}, 9, -5, p)^T$ and $\underline{v} = (\sqrt{2}, -5, -48, -19 + 3p)^T$.

10. Suppose that we are given a system of linear equations which is consistent and has a unique solution. If we change the numbers on the right-hand sides of the equations (and only those), then can it happen that the system of equations
a) has no solutions,
b) has infinitely many solutions?

11. (MT++'15) No matter how we omit one equation from a system of linear equations with four variables the system obtained will have a unique solution. At least how many equations should the original system of linear equations contain?

12. (MT+'16) A system of linear equations consisting of four equations has no such solutions for which the value of one of the variables is 0. Show that in this case there is a number $a \in \{1, 2, 3, 4, 5\}$ with the property that the system of linear equations has no such solutions in which the value of one of the variables is a either.

13. (MT++'16) A system of linear equations containing 5 unknowns has infinitely many solutions, but it has no such solutions for which the value of two of the variables are equal. Is it true that after the Gaussian elimination we always get a matrix with 4 nonzero rows?

14. (MT'17) Determine for which values of the parameter p the system of equations below is consistent. If it has solutions, then determine all of them.

$$
\begin{aligned}
x_1 + 3x_2 + 4x_3 &= 5 \\
2x_1 + 9x_2 + 14x_3 &= 13 \\
x_1 + p \cdot x_2 + p \cdot x_3 &= 4
\end{aligned}
$$

15. (MT+'17) Determine for which values of the parameter p the system of equations below is consistent. If it has solutions, then determine all of them.

$$
\begin{aligned}
x_1 + 3x_2 + 6x_3 + 2x_4 &= 5 \\
x_1 + 5x_2 + 8x_3 + 4x_4 &= 7 \\
2x_1 + 6x_2 + (p+12) \cdot x_3 + (p+5) \cdot x_4 &= 10
\end{aligned}
$$

16. (MT++'17) Determine for which values of the parameter p the system of equations below is consistent. If it has solutions, then determine all of them.

$$
\begin{aligned}
px_1 + 2px_2 + px_3 &= 3p \\
3x_1 + 9x_2 + 3x_3 &= 6 \\
x_1 + 2x_2 + 7x_3 &= 1 \\
3x_1 + 7x_2 + (p+15) \cdot x_3 &= 4
\end{aligned}
$$