Introduction to the Theory of Computing 1.

Exercise-set 6.

1. In $\mathbf{R}^4$ let $\underline{u} = (1,2,0,0)^T$, $\underline{v} = (0,1,2,0)^T$, $\underline{w} = (0,0,1,2)^T$, $\underline{a} = (1,1,0,4)^T$ and $\underline{b} = (1,1,1,4)^T$ (like in exercise 5/10.).
a) Determine whether $\underline{u}$, $\underline{v}$, $\underline{w}$ and $\underline{a}$ form a basis in $\mathbf{R}^4$ or not.
b) Determine whether $\underline{u}$, $\underline{v}$, $\underline{w}$ and $\underline{b}$ form a basis in $\mathbf{R}^4$ or not. If yes, then determine the coordinate vector of $\underline{a}$ in this basis.

2. Determine a basis in the following subspaces and determine their dimensions.
a) $V = \{(x,y,z)^T \in \mathbf{R}^3 : \; 5x - 6y + 9z = 0\}$,
b) $W = \{(x_1,x_2,x_3,x_4)^T \in \mathbf{R}^4 : \; 4x_1 + 3x_2 + 2x_3 + x_4 = 0\}$.

3. In the subspaces V below determine a basis containing the given vector $\underline{v}$, moreover determine the dimension of V.
a) V consists of those vectors in $\mathbf{R}^4$ in which the sum of the upper two coordinates is equal to the sum of the lower two coordinates, and $\underline{v} = (1,1,1,1)^T$.
b) V consists of those vectors in $\mathbf{R}^{100}$ whose coordinates form a geometric sequence with quotient 2 (from top down), and $\underline{v}$ is the one from these whose first coordinate is 7.

4. Let $\underline{u} = (2,3,4,4)^T$, $\underline{v} = (0,1,2,6)^T$, $\underline{w} = (0,0,1,1)^T$ and $\underline{a} = (8,9,11,-1)^T$.
Determine a basis in $\mathbf{R}^4$ containing the vectors $\underline{u}$, $\underline{v}$ and $\underline{w}$, then determine the coordinate vector of $\underline{a}$ in this basis.

5. (MT'16) Let the subspace V of $\mathbf{R}^4$ consist of those column vectors $\underline{x} \in \mathbf{R}^4$ for which $x_1 = x_2$ and $x_3 = 3x_4$ holds (where x_i denotes the ith coordinate of $\underline{x}$). Determine a basis in the subspace V and show that it is really a basis. (For the solution you don't need to show that V is in fact a subspace.)

6. (MT'12) Determine the dimension of the subspace of $\mathbf{R}^5$ given in exercise 5/6.

7. (MT'15) Let the subspace V of $\mathbf{R}^4$ consist of those column vectors $\underline{x} \in \mathbf{R}^4$ for which $x_1 - x_2 + x_3 = 0$ and $2x_1 + 3x_2 - x_3 + x_4 = 0$ holds. Determine the dimension of the subspace V. (Here $\underline{x} = (x_1,x_2,x_3,x_4)^T$, as usual. For the solution you don't need to show that V is in fact a subspace.)

8. (MT++'15) Let the set V consist of those vectors in $\mathbf{R}^5$ for which it holds that if we add an appropriate common number to each of their coordinates then we get a vector whose coordinates form a geometric progression with quotient 2. (E.g. the vector $(2,7,17,37,77)^T$ is like that, because if we add 3 to each of its coordinates we get a geometric progression with quotient 2.) Decide whether V forms a subspace in $\mathbf{R}^5$ or not. If yes, then determine the dimension of V.

9. (MT+'17) Let V be the subspace of $\mathbf{R}^4$ consisting of the vectors $\underline{x} = (x_1,x_2,x_3,x_4)^T$, whose coordinates satisfy the equation $x_4 = x_1 + 2x_2 - 3x_3$. Determine a basis in V containing the vector $\underline{v} = (1,1,1,0)^T$.

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10. Let $\underline{b}_1 = (2,4,1)^T$, $\underline{b}_2 = (3,-2,2)^T$ and $\underline{v} = (2,36,p)^T$.
For which values of the parameter p does $\underline{v} \in \langle \underline{b}_1, \underline{b}_2 \rangle$ hold? For this value of p determine the coordinate vector $[\underline{v}]_{\{\underline{b}_1,\underline{b}_2\}}$.

11. (MT'17) The vectors $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ form a basis of $\mathbf{R}^4$. Determine the dimension of the subspace generated by the vectors $\underline{a}+\underline{b}$, $\underline{c}+\underline{d}$, $\underline{a}+\underline{c}$, $\underline{b}+\underline{d}$.

12. (MT'07) The vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{100}$ form a basis of the subspace V of $\mathbf{R}^n$. What is the dimension of the subspace generated by the set of vectors $\{\underline{v}_1+\underline{v}_2, \underline{v}_2+\underline{v}_3, \ldots, \underline{v}_{99}+\underline{v}_{100}, \underline{v}_{100}+\underline{v}_1\}$?

13. (MT++'04) Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ be linearly independent vectors in $\mathbf{R}^n$. For which values of the parameter p does it hold that the vectors $\underline{a} = \underline{u} - \underline{v}$, $\underline{b} = \underline{u} + \underline{w}$, $\underline{c} = \underline{u} + \underline{v} - \underline{w}$ and $\underline{d} = p \cdot \underline{u} + \underline{v} + \underline{w}$ are linearly independent as well?

14. Suppose that in the subspace V of $\mathbf{R}^n$ the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ form a basis. For which values of the parameter p does it hold that the vectors $\underline{v}_1 - \underline{v}_2$, $\underline{v}_2 - \underline{v}_3, \ldots, \underline{v}_{k-1} - \underline{v}_k$, $\underline{v}_k - p \cdot \underline{v}_1$ form a basis in V as well?

15. (MT++'17) All that we know of the subspace V of $\mathbf{R}^4$ is that it contains each of the vectors $(1,0,1,0)^T, (1,0,0,1)^T, (0,1,0,1)^T$ and $(0,1,1,0)^T$. Can we determine the dimension of V from this?

16. (MT++'17) Let $\underline{a} = (1, 2, 4)^T$, $\underline{b} = (0, 1, 2)^T$ and $\underline{c} = (0, 0, 1)^T$ be vectors in $\mathbf{R}^3$.
a) Do the vectors $\underline{a}, \underline{b}, \underline{c}$ form a generating system in $\mathbf{R}^3$?
b) Do the vectors $\underline{a}, 3\underline{a} + \underline{b}, 6\underline{a} + 2\underline{b} + \underline{c}$ form a generating system in $\mathbf{R}^3$?

17. We know of the subspace V of $\mathbf{R}^n$ that $\dim V = k$, and the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ form a generating system in V. Is it always true, that in this case $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ form a basis in V?

18. (MT'14) Can we give four vectors in $\mathbf{R}^4$ in such a way that any two of them are linearly independent, but no three of them are linearly independent?

19. (MT+'14) Can we give five vectors in $\mathbf{R}^5$ in such a way that any three of them are linearly independent, but no four of them are linearly independent?

20. (MT++'14) Can we give five vectors in $\mathbf{R}^4$ in such a way that they form a generating system, but no four of them form a generating system?

21. The subspaces V and W of $\mathbf{R}^{99}$ both have dimension 50. Show that V and W have a common element which is not the zero vector.