Introduction to the Theory of Computing 1.

Exercise-set 5.

1. Let $V = \{\underline{u} = (u_1, u_2)^T \in \mathbf{R}^2 : u_1 \geq 0\}$ (the right halfplane). Is V a subspace of $\mathbf{R}^2$?

2. Determine whether the following subsets of $\mathbf{R}^4$ form a subspace or not:
a) those vectors in $\mathbf{R}^4$ all of whose coordinates are between 0 and 1;
b) those vectors in $\mathbf{R}^4$ in which the first coordinate equals the second.

3. Determine whether the following subsets of $\mathbf{R}^6$ form a subspace or not:
a) those vectors in $\mathbf{R}^6$ in which the coordinates are increasing (from the top down),
b) those vectors in $\mathbf{R}^6$ in which the sum of the upper three coordinates is the same as the sum of the lower three.

4. Let W_1 and W_2 be subspaces of $\mathbf{R}^n$. Are $W_1 \cup W_2$ and $W_1 \cap W_2$ (as sets of vectors) subspaces of $\mathbf{R}^n$ as well?

5. a) Do all the arithmetic sequences of length n form a subspace of $\mathbf{R}^n$?
b) And the geometric sequences?

6. (MT'12) Let's call a vector in $\mathbf{R}^5$ *Fibonacci-type* if all of its coordinates, starting form the third one, are sums of the previous two coordinates. (E.g. the vector $(3, -1, 2, 1, 3)^T$ is Fibonacci-type.) Do the Fibonacci-type vectors form a subspace in $\mathbf{R}^5$?

7. (MT+'15) Let the set W consist of the vectors $\underline{v} \in \mathbf{R}^5$ for which it holds that the difference of any two coordinates of $\underline{v}$ is an integer. (E.g. the vector $(3.6, 1.6, 4.6, 8.6, 0.6)^T$ is like that.) Decide whether W forms a subspace in $\mathbf{R}^5$ or not.

8. (MT+'16) Do the vectors $(x, y)^T$ for which $x^2 = y^2$ holds form a subspace of $\mathbf{R}^2$?

9. (MT++'16) Do the vectors $(x, y, z)^T$ for which $xy = yz$ holds form a subspace of $\mathbf{R}^3$?

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10. In $\mathbf{R}^4$ let $\underline{u} = (1, 2, 0, 0)^T$, $\underline{v} = (0, 1, 2, 0)^T$, $\underline{w} = (0, 0, 1, 2)^T$, $\underline{a} = (1, 1, 0, 4)^T$ and $\underline{b} = (1, 1, 1, 4)^T$.
a) Can $\underline{a}$ be written as a linear combination of $\underline{u}, \underline{v}$ and $\underline{w}$?
b) Can $\underline{b}$ be written as a linear combination of $\underline{u}, \underline{v}$ and $\underline{w}$?
c) Determine $\langle \underline{u},\ \underline{v},\ \underline{w} \rangle$, the subspace generated by $\underline{u}$, $\underline{v}$ and $\underline{w}$.
d) Determine $\langle \underline{u},\ \underline{v},\ \underline{w},\ \underline{a} \rangle$, the subspace generated by $\underline{u}$, $\underline{v}$, $\underline{w}$ and $\underline{a}$.
e) Determine $\langle \underline{u},\ \underline{v},\ \underline{w},\ \underline{b} \rangle$, the subspace generated by $\underline{u}$, $\underline{v}$, $\underline{w}$ and $\underline{b}$.

11. Determine the subspace of $\mathbf{R}^4$ spanned by $\underline{u} = (1, 1, 0, 0)^T$, $\underline{v} = (0, 1, 1, 0)^T$ and $\underline{w} = (0, 0, 1, 1)^T$.

12. We know of the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ that $\underline{v}_1$ is in the subspace generated by the other $n-1$ vectors, but none of the vectors $\underline{v}_2, \underline{v}_3, \ldots, \underline{v}_n$ is in the subspace generated by the other $n-1$ vectors. Prove that $\underline{v}_1 = \underline{0}$.

13. Determine the subspace generated by the vectors below. If that subspace is a line or a plane, determine its (system of) equation(s).
a) $(1, 0, 4)^T$, $(0, 1, -1)^T$,
b) $(2, -5, 1)^T$, $(-6, 15, -3)^T$,
c) $(3, 1, -4)^T$, $(4, 2, -3)^T$,
d) $(3, 1, -4)^T$, $(4, 2, -3)^T$, $(5, 3, -2)^T$.

14. (MT'08) Two vectors are given in 3-space, $\underline{a} = (2, 5, 1)^T$ and $\underline{b} = (1, -1, 3)^T$. Decide whether the subspace spanned by them is a line or plane and determine the equation of the geometric object obtained.

15. (MT++'15) Determine the subspace spanned by the following sets of vectors in $\mathbf{R}^3$. If the subspace is a line or plane, then determine its (system of) equation(s).
a) $(2, -6, 8)^T$, $(3, -9, 12)^T$,
b) $(2, -6, 8)^T$, $(3, -9, 11)^T$.

16. (MT'17) Determine the subspace generated by the vectors in $\mathbf{R}^3$ below. If that subspace is a line or a plane, determine its (system of) equation(s).

$$\underline{a} = (3, 1, 0)^T,\ \underline{b} = (5, 2, 1)^T,\ \underline{c} = (3, 2, 3)^T$$

17. (MT+'17) Let $\underline{u} = (0,0,1,2)^T$, $\underline{v} = (0,1,2,5)^T$ and $\underline{w} = (1,2,4,11)^T$ be vectors in $\mathbf{R}^4$. Determine $\langle \underline{u}, \underline{v}, \underline{w} \rangle$, the subspace generated by them. (That is, give a (system of) equation(s), satisfied by the vectors in $\langle \underline{u}, \underline{v}, \underline{w} \rangle$.)

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18. Let $\underline{a}, \underline{b}, \underline{c}$ be linearly independent vectors in $\mathbf{R}^n$. Prove that in this case the vectors $\underline{a}-\underline{b}$, $\underline{a}-\underline{c}$, $\underline{b}+\underline{c}$ are linearly independent as well.

19. MT+'09 Let $\underline{a}, \underline{b}, \underline{c}$ be linearly independent vectors in $\mathbf{R}^n$. Is it true that the vectors $\underline{a}+\underline{b}$, $\underline{b}+\underline{c}$, $\underline{c}+\underline{a}$ are linearly independent as well?

20. MT'16 Let $\underline{a}, \underline{b}, \underline{c}$ be linearly independent vectors in $\mathbf{R}^n$. Is it true that in this case the vectors $\underline{a}+\underline{b}+\underline{c}$, $\underline{a}+\underline{b}+3\underline{c}$, $3\underline{a}+\underline{b}+\underline{c}$ are linearly independent as well?

21. Let $\underline{a}, \underline{b}, \underline{c}$ be arbitrary vectors in $\mathbf{R}^n$ (for some n), and let $\underline{u} = \underline{a}+\underline{b}$, $\underline{v} = \underline{b}-\underline{c}$, $\underline{w} = \underline{c}+2\underline{a}$. Determine whether the following statements are true or not:
a) If $\underline{a}$, $\underline{b}$, $\underline{c}$ are linearly independent then $\underline{u}$, $\underline{v}$, $\underline{w}$ are linearly independent as well.
b) If $\underline{u}$, $\underline{v}$, $\underline{w}$ are linearly independent then $\underline{a}$, $\underline{b}$, $\underline{c}$ are linearly independent as well.

22. Show that every set of vectors in $\mathbf{R}^n$ containing the zero vector is linearly dependent.

23. Show that every set of vectors in $\mathbf{R}^n$ containing a vector twice is linearly dependent.

24. Prove that a subset of a linearly independent set of vectors in $\mathbf{R}^n$ is also linearly independent.

25. Let $\underline{a}_1, ..., \underline{a}_k$ be a linearly independent subset of $\mathbf{R}^n$ and let $\underline{x} = \sum_{i=1}^k \lambda_i \underline{a}_i$. Prove that $\underline{a}_1 \in \langle \underline{x}, \underline{a}_2, ..., \underline{a}_k \rangle$ holds if and only if $\lambda_1 \neq 0$.

26. MT'15 Let $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k, \underline{w} \in \mathbf{R}^n$ be arbitrary vectors. Suppose that $\underline{w} \neq \underline{0}$ and the set of vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{i-1}, \underline{v}_i + \lambda \cdot \underline{w}, \underline{v}_{i+1}, \ldots, \underline{v}_k$ is linearly independent for all the choices of the scalar $\lambda \in \mathbf{R}$ and the index $1 \leq i \leq k$. Is it true then that the set $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k, \underline{w}$ is also linearly independent?

27. MT++'15 Let $\underline{a}, \underline{b}$, and $\underline{c}$ be vectors in $\mathbf{R}^4$. Suppose that for any *integers* k, l and m not all of whose are 0, the linear combination $k \cdot \underline{a} + l \cdot \underline{b} + m \cdot \underline{c}$ is not the zero vector. Does it follow that $\underline{a}, \underline{b}, \underline{c}$ is a linearly independent set?

28. (MT'17) Suppose that the vectors $\underline{u}_1, \underline{u}_2, ..., \underline{u}_{10}$ in $\mathbf{R}^n$ are linearly dependent, but any 9 of them are linearly independent. Show that any linear combination of $\underline{u}_1, \underline{u}_2, ..., \underline{u}_{10}$ giving the $\underline{0}$ either all the coefficients are 0 or none of the coefficients are 0.(That is, show that if $c_1\underline{u}_1 + c_2\underline{u}_2 + ... + c_{10}\underline{u}_{10} = \underline{0}$ holds then either $c_1 = c_2 = \cdots = c_{10} = 0$ or $c_1 \cdot c_2 \cdot ... \cdot c_{10} \neq 0$.)

29. (MT+'17) Suppose that for the vectors $\underline{v}_1, \underline{v}_2, ..., \underline{v}_{10}, \underline{w}$ in $\mathbf{R}^n$ it holds that $\underline{v}_1, \underline{v}_2, ..., \underline{v}_{10}$ are linearly independent, but $\underline{v}_1, \underline{v}_2, ..., \underline{v}_{10}, \underline{w}$ are linearly dependent, and $\underline{w} \neq \underline{0}$. Show that there is an index $1 \leq i \leq 10$ and a scalar $\alpha \neq 0$, such that the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{i-1}, \underline{v}_i + \alpha \cdot \underline{w}, \underline{v}_{i+1}, \ldots, \underline{v}_{10}$ are linearly dependent.

30. (MT++'17) Determine whether the vectors $\underline{u} = (4,3,8,1)^T$, $\underline{v} = (2,0,4,0)^T$ and $\underline{w} = (3,5,6,2)^T$ in $\mathbf{R}^4$ are linearly independent or not.